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AN

ACADEMIC ALGEBRA

BY

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PREFACE.

THIS work is an attempt to present the subject of Elementary Algebra in a clear, scientific, and attractive form.

The order of subjects has been determined mainly by three considerations:

First, to introduce as early as possible the equation, of which Algebra is the science;

Second, to show how each subject bears directly on the end in view, namely, the solution and the use of equations;

Third, to eliminate as far as possible any unnecessary difficulties of the science.

Algebraic number is clearly defined and illustrated, its notation fully explained, and each principle proved once for all, whatever form it may afterwards assume. The distributive law and the law of quality are proved before they are used in addition and subtraction, as are also the principles of fractions before they are used in division. The chapter on Evolution is adapted to those who have not learned the extraction of roots in Arithmetic. Factoring is made fundamental in the study and solution of equations. The principles of equivalent equations and systems of equations are clearly proved, fully illustrated, and amply applied.

Fractions, ratios, and exponents are concisely and scientifically treated. Misleading old terms have been avoided, and useful new ones adopted.

The principles of limits, which are needed in Elementary Algebra and Geometry, are briefly and clearly presented.

The book is adapted to beginners of any age, and covers sufficient ground for admission to any American college or university. With the author's College Algebra it makes as extended a course as is usually taken in our best schools and colleges. The problems are varied, interesting, well graded, and not so difficult as to discourage the beginner. Very young pupils may omit some of the more difficult demonstrations in the first reading, but they should fully comprehend and become perfectly familiar with each principle and definition before new ones are considered. The best preparation for the constant application of the fundamental laws of number in Algebra is gained by carefully noting at the outset their use in all the operations with arithmetical number.

Any corrections or suggestions relating to the matter or method of the book will be thankfully received.

JAMES M. TAYLOR.

COLGATE UNIVERSITY,
May, 1893.

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ALGEBRA.

CHAPTER I.

ARITHMETICAL NUMBER AND ITS FUNDAMENTAL LAWS.

1. A **Quantity** is anything that can be increased, diminished, or measured.

For example, any distance, area, force, or weight is a quantity.

2. To **Measure** a quantity, we find how many times it contains some other quantity of the same kind taken as a **Unit,** or *standard of comparison.*

For example, to measure a distance, we find how many times it contains one foot, or one of the equal parts of a foot, according as we take a foot, or one of the equal parts of a foot, as the unit. To measure a period of time, we find how many times it contains an hour, or one of the equal parts of an hour, as the unit. From the idea of 2 times $\frac{1}{3}$ of an hour we obtain the idea of $\frac{2}{3}$ of an hour.

3. Arithmetical Number. In measuring quantities, we gain the idea of 'how many,' that is, of **arithmetical number.** If, in the measure of any quantity, we omit the unit of measure, we obtain an *arithmetical number.* It may be a whole number or a fraction.

Thus, by omitting the units ft., lb., hr., in 6 ft., 3 lbs., and $\frac{2}{3}$ hr., we obtain the whole numbers 6 and 3, and the fraction $\frac{2}{3}$.

Arithmetical numbers are generally represented by figures; but it is frequently found convenient to denote them by letters.

Thus in the subject of Interest, we often denote the *number* of dollars in the principal, interest, and rate per cent, by p, i, and r, respectively, and the *number* of years in the time by t. With this notation, the rule for finding the interest is concisely expressed by the equation $i = prt$.

The following signs are common to all branches of mathematics:

4. Signs of Operation. The sign $+$, read '*plus*,' indicates that the number after the sign is to be added to the number before it.

Thus $3 + 4 + 5$, read '3 plus 4 plus 5,' means that 4 is to be added to 3, and then 5 added to this result. So also $a + b$, read 'a plus b,' means that the number represented by b is to be added to the number represented by a; or, more briefly, it means that b is to be added to a.

The sign $-$, read '*minus*,' indicates that the number after the sign is to be *subtracted* from the number before it.

Thus $4 - 3$, read '4 minus 3,' means that 3 is to be subtracted from 4; $a + b - c$, read 'a plus b minus c,' means that b is to be added to a, and then c subtracted from this result.

The sign $\times$, or a point above the line, read '*multiplied by*,' or '*into*,' indicates that the number before it is to be multiplied by the number after it.

Thus 5×4, or $5 \cdot 4$, read '5 multiplied by 4,' or '5 into 4,' means that 5 is to be multiplied by 4; $a \times b \times c$, or $a \cdot b \cdot c$, means that a is to be multiplied by b, and then this result multiplied by c.

The sign of multiplication is usually omitted between two letters or a figure and a letter.

Thus $2ab$, read '$2ab$,' means $2 \times a \times b$; $7abc$, read '$7abc$,' means $7 \cdot a \cdot b \cdot c$. The sign of multiplication cannot be omitted between two factors when *both* are denoted by figures; for by the notation of arithmetic, 54 means $50 + 4$, and not 5×4.

The Sign of Division is $\div$, which is read '*divided by*,' or '*by*.'

Thus $a \div b$, read 'a divided by b,' or 'a by b,' means that the number a is to be divided by the number b; also $a \div b \times c \div d$, read 'a by b into c by d,' denotes that a is to be divided by b, the result multiplied by c, and then this result divided by d.

It should be noted that in a series of *additions and subtractions*, or in a series of *multiplications and divisions, the order of operations is from left to right.*

5. Signs of Relation. The Sign of Equality is $=$, which is read '*equals*,' or '*is equal to*.'

The Signs of Inequality are $>$ and $<$, which are read '*is greater than*,' and '*is less than*,' respectively, the opening being toward the greater number.

Thus $4 + 8 > 10$ is read '4 plus 8 is greater than 10'; and $4 + 2 < 12$ is read '4 plus 2 is less than 12.'

6. The Sign of Continuation is $\cdots$ or $---$, either of which is read '*and so on*,' or '*and so on to*.'

Thus $1, 2, 3, 4, \cdots$ is read '1, 2, 3, 4, and so on' indefinitely; $2, 4, 6, 8, \cdots, 32$ is read '2, 4, 6, 8, and so on to 32.'

The sign $\therefore$ stands for *hence* or *therefore*.

The sign $\because$ stands for *since* or *because*.

7. A **Mathematical Expression** is any symbol or combination of symbols that represents a number.

Thus, 4, $5 - 3$, $a + 5$, and $5a + 7x - 2b$ are mathematical expressions.

8. The **Signs of Grouping** are the parentheses ( ), the brackets [ ], the braces { }, the vinculum ——, and the bar |. Each sign of grouping indicates that the expression included by it is to be taken *as a whole.*

Thus each of the expressions $(7+5)\,9$, $[7+5]\,9$, $\{7+5\}\cdot 9$, $\overline{7+5}\times 9$, and $\left.\begin{array}{r}7\\+5\end{array}\right|9$, indicates that 5 is to be added to 7 and the result multiplied by 9. The sign of multiplication is understood between $(7+5)$ and 9, and $[7+5]$ and 9.

9. A **Term** is any expression in which the symbols of number are not connected by the sign + or −. Hence the parts of an expression which are connected by the sign + or − are its terms. In this definition an expression *within a sign of grouping* must be considered as a single symbol of number.

Thus each of the expressions 5, a, $3\,ab$, $5xy \div a$, and $7\,y \div x \times a \div c$ is a term; so also is $(5+3)\times 9$, $a \div (b+c)$, or $(a+b)\div(c+d)$. The expression $b + 2\,ax + 3\,b \div c$ consists of the three terms b, $2\,ax$, and $3\,b \div c$.

10. A **Monomial** is an expression of one term; as 4, $6\,xy$, $7\,bcx$, or $7(a+b)\div(x+y)$.

11. A **Polynomial** is an expression of two or more terms; as $4+7$ or $a+2\,xy+7\,b$.

A polynomial of two terms is called a **Binomial.**

A polynomial of three terms is called a **Trinomial.**

Thus $abx + ac \div b$ is a binomial, and $acx^2 + a \div c \times b + x \div y \div z$ is a trinomial.

A monomial is sometimes called a *simple expression,* and a polynomial a *compound expression.*

12. An **Equality** is a statement that two expressions represent the same number. The two expressions are called the **Members** of the equality.

Thus the equality $4\,x + 2 = 3\,x - 7$ states that the first member, $4\,x + 2$, represents the same number as the second member, $3\,x - 7$.

13. Remembering that all operations *within* each of two terms must be performed before the operations *between* them, it should be noted that the signs

$\times$ and $\div$ indicate *operations within a term;*

$+$ and $-$ indicate *operations between terms;*

while $=$, $>$, and $<$ indicate *relations of magnitude between expressions.*

EXAMPLE 1. Find the value of the expression,

$$9 \div (7-4) \times (5-3) - (12+4) \div 8.$$

$$\begin{aligned} 9 \div (7-4) \times (5-3) - (12+4) \div 8 &= 9 \div 3 \times 2 - 16 \div 8 \\ &= 6 - 2 = 4. \end{aligned}$$

The given expression consists of two terms.

EXAMPLE 2. Find the value of $5(2a-3b)-7bx$, when $a=5$, $b=2$, and $x=0$.

Putting 5 for a, 2 for b, and 0 for x, and writing the sign of multiplication where it is understood, we have

$$\begin{aligned} 5(2a-3b)-7bx &= 5(2 \times 5 - 3 \times 2) - 7 \times 2 \times 0 \\ &= 5(10-6) - 0 \\ &= 5 \times 4 = 20. \end{aligned}$$

Exercise 1.

Find the value of each of the following expressions:

1. $11+9-12-5$. 2. $5 \times 4 \div 2$. 3. $15 \times 8 \div (12 \div 2)$.

4. $4+8 \div 2 \times 3 - 12 \times 3 \div 9$.

5. $9 \div (5-2) \times (8-6) - (27-3) \div 8$.

If $a=1$, $b=2$, $c=3$, and $d=4$, find the value of

6. $c-b$. 8. $9b-5c$.

7. $2d-5a$. 9. $5d-10b$.

10. $17b-3d-5c+7a$. 14. $a \div c \times b \div d$.

11. $13a-6b+7c-5d$. 15. $b(a+5c-2d)$.

12. $4abc+3bcd-5ad$. 16. $(2b-3a)(3d-2c)$.

13. $5bcd+7ab-5abcd$. 17. $(a+b+c-d) \div (d-2a)$.

14. Addition is the process of finding the result when two or more expressions are united into one. The result, which must always be expressed in the simplest form, is called the **Sum**.

15. Law of Order. *Numbers to be added may be arranged in any order;* that is,

$$a + b = b + a. \qquad \text{(A)}$$

For let there be any two quantities of the same kind one containing a units and the other b units. Now if we put the second quantity with the first, the measure of the resulting quantity will be $a + b$ units; and if we put the first quantity with the second, the measure of the resulting quantity will be $b + a$ units. It is self-evident that these two resulting quantities will be equal; hence their measures will be equal;

$$\therefore a + b = b + a.$$

A similar proof would apply to an expression of any number of terms.

Thus $$\tfrac{1}{2} + 3 + \tfrac{1}{3} + 2 + \tfrac{1}{6} = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{6} + 3 + 2$$
$$= 1 + 3 + 2 = 6.$$

Here we change the order so as to add the fractions first.

NOTE. This law is often called the **Commutative Law of Addition.**

16. Law of Grouping. *Numbers to be added may be grouped in any manner;* that is,

$$a + b + c = a + (b + c). \qquad \text{(B)}$$

For by the *law of order* we have

$$\begin{aligned} a + b + c &= b + c + a \\ &= (b + c) + a = a + (b + c). \end{aligned}$$

A similar proof would apply in any other case.

Thus $$\begin{aligned} 3 + 5 + \tfrac{2}{3} + \tfrac{1}{4} + \tfrac{1}{12} &= 3 + 5 + (\tfrac{2}{3} + \tfrac{1}{4} + \tfrac{1}{12}) \\ &= 3 + 5 + 1 = 9. \end{aligned}$$

Here we group the fractions together, add them, and then add their sum to the whole numbers.

NOTE. This law is often called the **Associative Law of Addition.**

The use of the laws of order and grouping in arithmetic may be illustrated at will, as by the addition of 45 and 23; thus

$$\begin{aligned} 45 + 23 &= 40 + 5 + 20 + 3 \\ &= 40 + 20 + 5 + 3 && \text{by (A)} \\ &= (40 + 20) + (5 + 3) && \text{by (B)} \\ &= 60 + 8 = 68. \end{aligned}$$

Writing one number under the other and then adding the vertical columns, as we do in arithmetic, is but a convenient way of performing the operations indicated above.

17. Subtraction is the process of taking from one number, called the **Minuend,** another number called the **Subtrahend.** This result, which must be expressed in the simplest form, is called the **Remainder.**

The subtrahend and the remainder are evidently the two parts of the minuend; hence, since the whole is equal to the sum of all its parts, we have

$$\text{minuend} = \text{subtrahend} + \text{remainder}.$$

18. Multiplication. As first defined in arithmetic, to multiply one number by another is to take the first number as many times as there are units in the second. Each of the two numbers is called a **Factor,** and the result a **Product.** The first factor is called the **Multiplicand,** and the second the **Multiplier.**

By the definition given above we can multiply only by whole numbers. To include fractions and algebraic numbers as multipliers we need a more general definition. We define a product therefore as follows:

19. The **Product** of two numbers is the result obtained by treating the multiplicand in the same way that we would treat 1 to obtain the multiplier.

First consider the product of 4×3.

To obtain the multiplier 3 from 1, we take 1 three times; that is, $3 = 1 + 1 + 1$.

Hence to obtain the product of 4×3, we take 4, the multiplicand, three times; that is, $4 \times 3 = 4 + 4 + 4$.

Next consider the product of $9 \times \frac{2}{3}$.

To obtain the multiplier $\frac{2}{3}$ from 1, we divide 1 by 3 and multiply the result by 2; that is, $\frac{2}{3} = 1 \div 3 \times 2$.

Hence to obtain the product of $9 \times \frac{2}{3}$, we divide 9, the multiplicand, by 3 and multiply the result by 2; that is, $9 \times \frac{2}{3} = 9 \div 3 \times 2$.

20. Law of Order. *Factors may be arranged in any order;* that is,

$$ab = ba. \qquad (A')$$

When a and b *are integers,* write down b rows of dots, putting a dots in each row, as below:

● ● ● ● ● ● ⋯ a in a row
● ● ● ● ● ● ⋯
● ● ● ● ● ● ⋯
⋯ ⋯ ⋯ ⋯ ⋯ ⋯ ⋯

b rows.

Counting by rows, the whole number of dots is a taken b times, that is $a \times b$; counting by columns, the whole number of dots is b taken a times, that is $b \times a$. Hence $a \times b$ and $b \times a$ equal the same number of dots.

$$\therefore ab = ba \qquad (1)$$

Suppose that a and b are fractional, and that $a = \frac{3}{7}$ and $b = \frac{4}{5}$; then by § 19

$$ab = \tfrac{3}{7} \times \tfrac{4}{5} = \tfrac{3}{7} \div 5 \times 4$$

$$= \frac{3 \times 4}{7 \times 5}, \qquad (2)$$

and
$$ba = \tfrac{4}{5} \times \tfrac{3}{7} = \tfrac{4}{5} \div 7 \times 3$$
$$= \frac{4 \times 3}{5 \times 7} \text{ or } \frac{3 \times 4}{7 \times 5}. \qquad (3)$$

From equalities (2) and (3), ab and ba are equal to the same expression,

$$\therefore ab = ba.$$

The same reasoning would apply to any other fractional values of a and b; hence (A′) holds true for all values of a and b.

NOTE. This law is often called the **Commutative Law of Multiplication.**

21. Law of Grouping. *Factors may be grouped in any manner;* that is,

$$abc = a(bc). \qquad (\text{B}')$$

For by the *law of order* we have

$$abc = bca$$
$$= (bc)a = a(bc).$$

NOTE. This law is often called the **Associative Law of Multiplication.**

EXAMPLE. Multiply $3\,ac$ by $7\,b$.

$$(3\,ac) \times (7\,b) = 3 \cdot a \cdot c \cdot 7 \cdot b \qquad \text{by } (\text{B}')$$
$$= 3 \cdot 7 \cdot a \cdot b \cdot c \qquad \text{by } (\text{A}')$$
$$= 21\,abc.$$

22. Distributive Law. *The product of two expressions is equal to the sum of the products obtained by multiplying each term of either expression by the other;* that is,

$$(a + b + c)x = ax + bx + cx. \qquad (\text{C})$$

Law (C) is easily proved when x is any whole number.

Thus
$$(a + b) \times 3 = (a + b) + (a + b) + (a + b)$$
$$= a + a + a + b + b + b \qquad \text{by (A), (B)}$$
$$= a \times 3 + b \times 3 = 3\,a + 3\,b.$$

The general proof of Law (C) will be given in § 54.

NOTE. By Law (C) multiplications may be distributed over additions and subtractions, hence it is called the **Distributive Law of Multiplication.**

The use of Law (C) in arithmetic may be illustrated at will, as by the multiplication of 42 by 3; thus,

$$\begin{aligned} 42 \times 3 &= (40 + 2) \times 3 \\ &= (40 \times 3) + (2 \times 3) \qquad \text{by (C)} \\ &= 120 + 6 = 126. \end{aligned}$$

Exercise 2.

Find the product of

1. $7ab$ and $5cx$.
2. $2a$, $5b$, and $7x$.
3. $3x$, $4y$, and $7a$.
4. $2ab$ and $3xy$.
5. $4xy$ and $\frac{1}{2}abc$.
6. $3x$, $\frac{1}{2}y$, and $\frac{1}{4}c$.
7. $6a$, $\frac{2}{3}b$, and $\frac{3}{4}x$.
8. $\frac{1}{2}a$, $\frac{1}{3}x$, and $\frac{5}{6}y$.
9. Prove that $(a+b+c) \times 3 = 3a + 3b + 3c$.
10. Prove that $(a+b) \times 5 = 5a + 5b$.
11. Show how the distributive law is used in multiplying 24 by 2; 231 by 3; 421 by 4; 2314 by 2.

23. Division. Having given a product and one factor, **Division** is the process of finding the other factor. The given product is called the **Dividend**; the given factor, the **Divisor**; and the required factor, the **Quotient.**

From their definitions, the divisor and quotient are evidently the two factors of the dividend; hence, since any number is equal to the product of its factors, we have

$$\text{quotient} \times \text{divisor} = \text{dividend}. \qquad (1)$$

Let D denote the dividend, and d the divisor; then the expression $D \div d$ will denote the quotient, and by (1) we shall have

$$(D \div d) \times d = D. \qquad (2)$$

CHAPTER II.

EQUATIONS AND PROBLEMS.

24. Coefficients. If a term be resolved into two factors, either is called the **Coefficient,** or the **Co-factor,** of the other. The factor expressed in figures is called the **Numerical Coefficient** of the other factor.

Thus in $4\,abc$, 4 is the *numerical coefficient* of abc, $4\,a$ is the coefficient of bc, and $4\,ab$ the coefficient of c. When no numerical coefficient is written, 1 is understood; thus a denotes $1\,a$, abc denotes $1\,abc$.

25. Like or **Similar Terms** are such as do not differ, or differ only in their coefficients.

Thus $4\,abc$ and $4\,abc$ are like terms; so also are $4\,abc$ and $10\,abc$. Again, $6\,axy$ and $4\,bxy$ are similar terms, if we regard $6\,a$ and $4\,b$ as the coefficients respectively; but if 6 and 4 be taken as the coefficients, these terms are *dissimilar*.

Terms that are not similar are said to be **Unlike** or **Dissimilar**; as $4a$, $3\,b$, $7\,cd$.

26. Similar terms are added, or united into one term, by the Distributive Law; for by (C) of § 22 we have

$$ax + bx + cx = (a + b + c)\,x.$$

Example 1. Find the sum of $2\,x$, $5\,x$, $7\,x$, and $4\,x$, x denoting any arithmetical number.

The sum is denoted by $2\,x + 5\,x + 7\,x + 4\,x$, and by law (C)

$$\begin{aligned}2\,x + 5\,x + 7\,x + 4\,x &= (2 + 5 + 7 + 4)\,x\\ &= 18\,x.\end{aligned}$$

Example 2. Find the sum of xy, $3xy$, axy, and $6xy$, a, x, and y denoting any arithmetical numbers.

$$\begin{aligned} xy + 3xy + axy + 6xy &= 1(xy) + 3(xy) + a(xy) + 6(xy) && \text{by (B)} \\ &= (1 + 3 + a + 6)xy && \text{by (C)} \\ &= (10 + a)xy. \end{aligned}$$

Exercise 3.

Find the sum of

1. $2x$, $7x$, and $9x$.
2. y, $3y$, $6y$, and $2y$.
3. $\frac{1}{2}b$, $\frac{2}{3}b$, and $\frac{3}{4}b$.
4. $\frac{1}{5}a$, $\frac{1}{3}a$, and $\frac{7}{15}a$.
5. $2ax$, $5ax$, and $9ax$.
6. abc, $9abc$, and $14abc$.
7. $\frac{1}{3}cx$, $\frac{1}{4}cx$, and $\frac{1}{2}cx$.
8. $2xy$, $3xy$, and axy.
9. $\frac{1}{2}xy$, $5xy$, $\frac{2}{3}xy$, $7xy$, $\frac{5}{6}xy$, and $16xy$.
10. xyz, $axyz$, $5xyz$, $bxyz$, and $7xyz$.
11. $3abc$, $5abc$, $\frac{1}{2}abc$, $\frac{2}{3}abc$, $4abc$, and $\frac{3}{5}abc$.
12. If one part of 40 is x, what is the other part?
13. If the difference of two numbers is 15 and the smaller is x, what is the greater?
14. How far can a man walk in x hours at the rate of 3 miles an hour? At the rate of b miles an hour?
15. How long will it take a man to walk x miles at the rate of 3 miles an hour? At the rate of b miles an hour?
16. How old will a man be in c years, if his present age is x years?
17. If one part of 4 is x, what is the other part?
18. If one part of a is x, what is the other part?
19. If $60 is divided equally among x boys, how much will each boy receive?
20. In b years a man will be x years old. What is his present age?

21. If one number is x, and another is 3 times as great, what is the second number?

22. If a goat costs x dollars, and a cow costs 4 times as much as a goat, and a horse costs 3 times as much as a cow, how much does a horse cost?

23. A has x dollars, B has twice as many dollars as A, and C has as many as A and B together. How many dollars has C?

24. A is x years old, B is 7 years older than A, and C's age equals the sum of A's and B's ages. How old is C?

25. A travelled $2x$ miles, B 4 times as far, and C three-fifths as far as A and B together. How far did C travel?

26. If 10 sheep cost x dollars, and 5 cows cost $2x$ dollars, what would be the cost of 15 sheep and 7 cows?

27. Known and Unknown Numbers. Any problem involves both known and unknown numbers. For the sake of distinction, a known number that is not expressed by figures is represented by one of the *first* letters of the alphabet, a, b, c, etc.; and an unknown number, that is, one to be found, is represented by one of the *last* letters of the alphabet, x, y, z, etc.

28. An **Axiom** is a self-evident truth.

The axioms most frequently used in Algebra are the following:

1. If the same number or equal numbers be added to equal numbers, the sums will be equal.
2. If the same number or equal numbers be subtracted from equal numbers, the remainders will be equal.
3. If equal numbers be multiplied by the same number or equal numbers, the products will be equal.
4. If equal numbers be divided by the same number, except *zero*, or by equal numbers, the quotients will be equal.

5. Numbers which are equal to the same number or to equal numbers are equal to each other.

29. Identical Expressions. Equal expressions that contain only figures, or expressions that are equal for all values of their letters, are called **Identical Expressions.**

Thus $4 + 6$ and 5×2 are identical expressions; so also are $6 - 3 \times 2$ and 0. Again, ab and ba are identical expressions; so also are $(a + b)x$ and $ax + bx$; for these expressions are equal for all values of their letters a, b, and x.

30. Identities and Equations. Equalities are of two kinds, *Identities* and *Equations.*

The statement that two identical expressions are equal is called an **Identity.**

In writing identities the sign $\equiv$, read 'is identical with,' is often used instead of the sign $=$.

Thus each of the equalities $4 + 6 = 5 \times 2$, $ab = ba$, and $(a + b)x = ax + bx$ is an identity; for each holds true for all values of its number symbols. To indicate that these equalities are identities, they may be written $4 + 6 \equiv 5 \times 2$, $ab \equiv ba$, and $(a + b)x \equiv ax + bx$.

If two expressions are not identical, and one or both of them contains a letter or letters, the statement that they are equal is called an **Equation.**

Thus the equalities $3x = 6$ and $2x - 5 = 7$ are equations; for evidently $3x$ and 6 are not identical expressions, nor are $2x - 5$ and 7.

The expressions $3x$ and 6 are equal when $x = 2$; for if in the equation $3x = 6$ we put 2 for x, we obtain the identity $3 \times 2 = 6$; but if we put any other number as 3 for x, we obtain an absurdity, as $3 \times 3 = 6$. The expressions $2x - 5$ and 7 are equal when $x = 6$; for if in the equation $2x - 5 = 7$ we put 6 for x, we obtain the identity $2 \times 6 - 5 = 7$; but if we put any other number as 5 for x, we obtain an absurdity, as $2 \times 5 - 5 = 7$.

Hence an equation holds true only for special values, or sets of values, of its letters.

31. A **Solution** of an equation of one unknown number, as x, is any value of x for which the equation is true. When a solution of an equation in x is substituted for x, the equation becomes an identity, and is said to be *satisfied.*

Thus a solution of the equation $2x - 5 = x + 7$ is 12; for putting 12 for x, we obtain the identity $24 - 5 = 12 + 7$.

NOTE. The word *solution* denotes either the *process* of solving, or the *result* obtained by solving. Here it is used in the latter sense. A solution of an equation of one unknown number is often called a **Root** of the equation.

32. To **Solve** an equation is to find its solutions.

The process by which we solve the simplest forms of equations is illustrated by the following examples:

EXAMPLE 1. Solve the equation $7x + 5 = x + 23$. (1)

Subtract x from each member, $6x + 5 = 23$. Ax. 2

Subtract 5 from each member, $6x = 18$. Ax. 2

Divide each member by 6, $x = 3$. Ax. 4

Whence we know that any value of x that will satisfy equation (1) will satisfy also each of the other three equations. Hence if equation (1) has any solution, it is 3.

PROOF. Putting 3 for x in equation (1), we obtain the identity

$$21 + 5 = 3 + 23;$$

hence 3 is a solution of equation (1).

In solving equation (1) we put all the terms involving the unknown number in one member and the known terms in the other member. What operations are necessary to do this in the following simple equations the student can easily determine.

Example 2. Solve the equation $18x - 7 = 18 - 7x$. (1)

Add $7x$, $25x - 7 = 18$. Ax. 1
Add 7, $25x = 25$. Ax. 1
Divide by 25, $x = 1$. Ax. 4

Hence if equation (1) has any solution, it is 1.

Proof. Substituting 1 for x in equation (1), we obtain the identity $18 - 7 = 18 - 7$; hence 1 is a solution of equation (1).

Exercise 4.

Solve each of the following equations:

1. $3x - 7 = 2x + 3$.
2. $3x + 4 = x + 10$.
3. $4x + 4 = x + 7$.
4. $7x + 5 = x + 23$.
5. $8x = 5x + 42$.
6. $6x - 5 = 4x + 1$.
7. $18x - 7 = 43 - 7x$.
8. $5x - 7 = 3x + 1$.
9. $19x - 11 = 15 + 6x$.
10. $3x + 15 = x + 25$.
11. $5x - 2 = 3x + 4$.
12. $7x - 9 = 17 + 2x$.
13. $\frac{2}{3}x - 4 = 5 - \frac{1}{3}x$.
14. $\frac{4}{5}x - 3 = 7 - \frac{1}{5}x$.
15. $\frac{7}{8}x - \frac{1}{2} = \frac{1}{3} - \frac{1}{8}x$.
16. $\frac{3}{4}x - \frac{1}{5} = \frac{5}{6} - \frac{1}{4}x$.
17. $3(4x + 7) + 5 = 50$.
18. $6x + 3(4x + 3) = 41$.
19. $6(x + 2) = 4(x + 3) + 15$.
20. $3(2x + 4) = 4(x + 5) + 32$.

33. Problems solved by Equations. For solving problems by equations, no general rule can be given. The following suggestions, however, may prove helpful in solving the problems that follow:

Carefully study each problem until it is clearly understood.

Let x denote the unknown number; or, if there are two or more unknown numbers, let x or some multiple of x denote one of them, and then express each of the others in terms of x.

Form an equation expressing the relations given in the problem. Then solve this equation.

Example 1. The sum of two numbers is 80, and the greater is 3 times the less. What are the numbers?

Let $x =$ the less number;

then $3x =$ the greater number.

Hence by the conditions, $x + 3x = 80.$ (1)

Add like terms, $4x = 80.$

Divide by 4, $x = 20$, the less number.

Multiply by 3, $3x = 60$, the greater number.

That is, the less number is 20, and the greater 60. Substituting 20 for x in (1), we obtain the identity $20 + 60 = 80$.

Example 2. A farmer bought a horse, a cow, and a goat; the horse cost 3 times as much as the cow, and the cow 4 times as much as the goat, and all three together cost $255. What was the cost of each?

Let $x =$ the *number* of dollars the goat cost;

then $4x =$ the *number* of dollars the cow cost,

and $12x =$ the *number* of dollars the horse cost.

Hence, by the conditions, $x + 4x + 12x = 255.$

Add like terms, $17x = 255.$

Divide by 17, $x = 15.$ Ax. 4

Multiply by 4, $4x = 60.$ Ax. 3

Multiply by 3, $12x = 180.$ Ax. 3

Hence the goat cost $15, the cow $60, and the horse $180.

Exercise 5.

1. A line 30 inches long is divided into two parts, one of which is double the other. How long are the parts?

Ans. 20 and 10 inches.

2. A, B, and C together have $90. B has twice as much as A, and C has as much as A and B together. How much has each?

Let x = the *number* of dollars A has;
then $2x$ = the *number* of dollars B has;
hence $3x$ = the *number* of dollars C has.
$\therefore x + 2x + 3x = 90.$

3. The sum of the ages of A and B is 67 years, and A is 17 years older than B. What is the age of each?
Ans. 42 and 25 years.

4. Three men, A, B, and C, trade in company and gain $600, of which A is to have 3 times as much as B, and C as much as A and B together. What is the share of each?

Let x = the number of dollars B is to have, etc.

5. A farmer bought 3 cows for $180, and the prices paid were as the numbers 1, 2, and 3. What was the cost of each?

Let x = the *number* of dollars paid for the first;
then $2x$ = the *number* of dollars paid for the second,
and $3x$ = the *number* of dollars paid for the third.

6. Divide 500 into two parts which are as the numbers 1 and 4.

7. What number is that whose double exceeds its half by 27?

8. Divide $575 between A and B so that A may receive $75 more than B.

Let x = the *number* of dollars B receives;
then $x + 75$ = the *number* of dollars A receives;
hence $2x + 75 = 575.$ (1)

9. Divide 105 into two parts whose difference is 45.

10. What number is that to which if 40 be added the sum will be 3 times the original number?

11. Divide $84 between A, B, and C, so that B shall have $13 more than A, and C $16 more than B.

12. Three men, A, B, and C, contribute to an enterprise $2400. B put in twice as much as A, and C put in as much as A and B together. How much did each contribute?

13. Find two numbers whose difference is 10, and one of which is 3 times the other.

14. A man 40 years old has a son 10 years old. In how many years will the son be $\frac{1}{3}$ as old as the father?

15. A father's age is 3 times that of his son, and in 10 years it will be twice as great. How old is each?

16. One man is 70 years of age, and another 50. When was the first twice as old as the second?

17. If two men, 150 miles apart, travel toward each other, one at the rate of 2 miles an hour, and the other at the rate of 3 miles an hour, in how many hours will they meet?

18. A horse, carriage, and harness together are worth $625. The horse is worth 8 times as much as the harness, and the carriage is worth $125 more than the harness. Find the value of each.

Ans. $400, $175, and $50.

19. A man bought a cow, a sheep, and a hog for $80; the cow cost $32 more than the sheep, and the sheep $6 more than the hog. Find the price of each.

Ans. $50, $18, $12.

20. The sum of $6000 was divided between A, B, C, and D; B received twice as much as A, C as much as A and B together, and D as much as A, B, and C together. How much did each receive?

Ans. $500, $1000, $1500, $3000.

21. A farmer spent $480 in buying cows and sheep. Each cow cost $30, and each sheep $6, and the number of animals bought was 40. How many cows did he buy, and how many sheep?

Let $x =$ the number of cows bought.

22. A man has two sons and one daughter. He wishes to divide $12,000 among them so that the younger son shall have twice as much as the daughter, and the older son as much as both the other children. How much must he give to each?

23. Divide 90 into five parts so that the second shall be 5 times the first, the third shall be $\frac{2}{3}$ of the first and second, the fourth shall be $\frac{1}{2}$ of the first, second, and third, and the fifth shall be 2 times the sum of the other four.

24. A, B, and C enter into partnership to do business. A furnishes 5 times as much capital as B, and C furnishes $\frac{1}{2}$ as much as A and B together. They all together furnish $18,900. How much does each furnish?

25. A gentleman, dying, bequeathed his property of $21,840 as follows: to his son 2 times as much as to his daughter, and to his widow $1\frac{1}{3}$ times as much as to both his son and daughter. What was the share of each?

26. A farmer purchased 100 bushels of grain. He bought 2 times as many bushels of corn as of oats, and $2\frac{1}{3}$ times as many bushels of wheat as of oats and corn. How many bushels of each kind did he buy?

27. Three candidates for an office polled the following votes respectively: B received 3 times as many votes as A, and C $1\frac{1}{2}$ times as many as A and B together. The whole number of votes was 11,000. How many votes did each receive?

28. A banker loaned to each of 4 men equal sums of money. One man had the money 2 years, another $2\frac{1}{2}$ years, another $3\frac{1}{2}$ years, and another $4\frac{1}{2}$ years. The entire interest money received was \$275. How much did each man pay?

Let $x =$ the number of dollars in the yearly interest on the sum loaned to each man.

29. A library contains 9 times as many historical works, and 5 times as many scientific books, as works of fiction. The historical works exceed the works of fiction and science by 10,500 volumes. How many volumes are there of each?

30. A drover, being asked how many sheep he had, replied that if he had 3 times as many as he then had and 6 more, he would have 150. How many had he?

31. The expenses of a manufacturer for 5 years were \$17,500. If they increased \$500 annually, what were his yearly expenses?

32. A farmer had 590 sheep distributed in three fields. In the first field there were 25 more than in the second, and in the third there were 15 more than in the first. How many sheep were in each field?

33. Two pedestrians travelled toward each other at the rate of 4 miles per hour. When they met they discovered that one had travelled 2 hours longer than the other, and that the entire distance travelled by both was 72 miles. How far did each travel?

Let $x =$ the number of hours the second pedestrian travelled.

34. Five boys bought a bicycle, but two of them failing to pay their shares, each of the others had to pay \$16 more. What was the price of the bicycle?

34. Algebra is that branch of mathematics which treats of the *equation*, its nature, the methods of solving it, and its use as an instrument for mathematical investigation.

Each of the equations thus far considered has an arithmetical solution; that is, an arithmetical number satisfies each of them. But many equations have no arithmetical solution; for example, take the equation

$$3x - 7 = 2x - 14. \quad (1)$$

Add 7, $$3x = 2x - 7. \quad (2)$$

Subtract $2x$, $$x = -7. \quad (3)$$

Hence if (1) has any solution, it is -7. But -7 has no meaning in arithmetic. In fact, in arithmetic, $2x$ could not be subtracted from both members of equation (2) to obtain (3); for the second member of (2) is evidently less than $2x$, and in arithmetic we cannot subtract a greater number from a less. Hence to solve equations in general, we must so enlarge our idea of number as to give a meaning to such expressions as -7, and to enable us to subtract a greater number from a less.

Such expressions as -7 represent *algebraic numbers*, which have *quality* as well as *arithmetical value*. Hence preparatory to the further study and use of the equation, we must gain a knowledge of *algebraic number*, its nature, its notation, and its laws of combination. These subjects will be treated in the next chapters.

CHAPTER III.

ALGEBRAIC NUMBER AND ITS FUNDAMENTAL LAWS.

35. Positive and Negative Quantities. Two quantities of the same kind are opposite in quality, if when added any amount of the one destroys an equal amount of the other. Of two quantities, opposite in quality, one is said to be **Positive** in quality, and the other **Negative.**

Thus credits and debits are opposite in quality, since when added any amount of either destroys an equal amount of the other; as for example, $15 of debt destroys $15 of credit. If we call credits positive, debits will be negative. The energies of two boys pulling in opposite directions are opposite in quality; since, when added, equal amounts of the two destroy each other; if we call one positive, the other will be negative.

Distances to the right are the opposites of distances to the left; for if a movement of 4 inches to the right be followed by a movement of 4 inches to the left, no advance will have been made in either direction. Also distances upward are the opposites of distances downward. If we call distances to the right or upward positive, then distances to the left or downward will be negative.

The sign +, read *'positive,'* and the sign −, read *'negative,'* are often used with the measures of quantities to denote their *quality* as positive or negative.

Thus if we call credit positive, + $5 will denote $5 of credit, and −$4 will denote $4 of debt. If +8 inches denotes 8 inches to the right, −9 inches will denote 9 inches to the left. If $+3^\circ$ denotes 3° above the zero point, -7° will denote 7° below that point. If +400 years denotes 400 years after Christ, −300 years will denote 300 years before Christ.

36. Algebraic Number. By omitting the particular units \$, and °, in + \$15, − \$4, $+3°$, and $-7°$, we obtain the algebraic numbers, $+15$, -4, $+3$, and -7. $+5$ is read 'positive 5'; and -4 is read 'negative 4.' Each of these numbers has not only an *arithmetical value,* but also the *quality* of one of two opposites; hence

An *Algebraic Number* is one that has both an *arithmetical value* and the *quality* of one of two opposites.

Thus $+5$ and -5 are opposite in quality; but they have the same *arithmetical,* or *absolute, value;* for $+5=(+1)\times 5$, and $-5=(-1)\times 5$.

Any two algebraic numbers that are equal in arithmetical value but opposite in quality destroy each other when added.

Thus the sum of $+5$ and -5 is zero.

37. Symbols of Number. Algebraic numbers are denoted by figures with the sign + or − prefixed to denote their quality, or by letters. When no sign of quality is written before a term, the sign + is understood.

Figures alone denote arithmetical numbers; hence, when figures are used, to express *fully* an algebraic number we must prefix the sign + or − to denote its quality. Thus each of the expressions $+3$, -7, -1, $+8$ denotes an algebraic number, the figure denoting the arithmetical value, and the sign the quality. The sign of quality, however, is usually omitted before figures that denote the arithmetical value of a positive number. Thus, as an algebraic expression, 4 denotes $+4$, 17 denotes $+17$, $3a$ denotes $+3a$, and x denotes $+1x$. As an arithmetical expression, 4 denotes a number without quality.

A letter represents both the *arithmetical value* and the *quality* of an *algebraic number.*

Thus a denotes either a positive or a negative number, so also does b, x, y, or z; each letter denotes both the arithmetical value and the quality of an algebraic number.

In this chapter we shall generally denote algebraic numbers by figures with signs prefixed. This will aid the student to fix in mind the real nature of algebraic number before he begins the general use of the literal notation.

The sign before a term containing letters denotes the quality of its numerical coefficient.

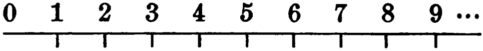
Thus in the term $-4a$, the sign $-$ denotes the quality of 4, and not that of a or of the whole term; a may denote $+4$, -4, -7, $+18$, or any other number. In any connected series of operations, any given letter must throughout represent the same number.

When we wish to indicate the quality of a number and either addition or subtraction at the same time, we enclose the number symbol and its quality sign within parentheses.

Thus $(-3)+(+1)$, read 'negative 3 plus positive 1,' denotes that $+1$ is to be added to -3. The sum is evidently -2.

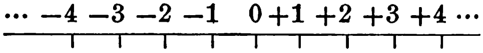
For $(-3)+(+1)=(-2)+(-1)+(+1)=-2$.

38. Range of Algebraic Number. The element of quality in algebraic number makes the range of such number double that of arithmetic. Thus the integers of arithmetic make up the single series,



(1)

The integers of algebra extending in both directions from zero make up the double series,



(2)

If the divisions of the lines in (1) and (2) be taken as units of length, then each number in (1) denotes simply its distance from the zero point; while each number in (2) denotes not only its distance, but also its direction, from the zero point.

NOTE. Arithmetical numbers must not be regarded as positive numbers. An arithmetical number has no quality.

39. An algebraic number is said to be *increased by adding to it a positive number.* If in series (2) of § 38 we add $+1$ to any number, we obtain the next right-hand number. Thus

$$(-4)+(+1)=-3,$$
$$(-2)+(+1)=-1,$$

and so on for the entire series; that is, the numbers in series (2) increase from left to right.

Hence a *positive number is the greater, the greater its arithmetical value; while a negative number is the greater, the less its arithmetical value.*

Thus $+4>+3$ and $+3>+2$;
but $-4<-3$ and $-3<-2$.

40. Two Uses of the Signs + and —. The use of the signs + and — to indicate quality must be carefully distinguished from their use to denote operations. The sign before a monomial or before the first term of any polynomial is *always* a sign of quality. As will be shown hereafter, any *algebraic* expression will denote the same number, whether we regard the signs connecting its terms as signs of quality or of operation. In general formulas, such as (A), (B), and (C), it is of advantage to regard the signs + and — as signs of operation; but in most other cases it is better to regard the sign written between any two terms of an expression as a sign of quality, the sign of addition being understood between each two consecutive terms.

Thus $5-3$ denotes $(+5)+(-3)$, or the sum of the terms $+5$ and -3; and $7ac-4x+3y$ denotes $(+7ac)+(-4x)+(+3y)$, or the sum of the terms $+7ac$, $-4x$, and $+3y$.

ADDITION AND SUBTRACTION.

41. Laws of Addition. The demonstrations given in §§ 15 and 16 evidently hold for algebraic numbers; hence both the law of order and the law of grouping hold for algebraic addition.

Both of these laws are illustrated in finding how much an estate is worth, for we can take into account the different items of property (credits being +, and debts –) in any *order* or in any *groups* we please.

42. Rules for Addition. (i.) To add two or more numbers of the same quality, *find the sum of their arithmetical values and prefix to it their common sign.*

For example, to add + 4, + 7, and + 11, we have

$$\begin{aligned}(+4)+(+7)+(+11) &= (+1)\cdot 4+(+1)\cdot 7+(+1)\cdot 11\\ &= (+1)\cdot 22 = +22.\end{aligned}$$

Again, to add – 7, – 2, and – 15, we have

$$\begin{aligned}(-7)+(-2)+(-15) &= (-1)\cdot 7+(-1)\cdot 2+(-1)\cdot 15\\ &= (-1)\,24 = -24.\end{aligned}$$

(ii.) To add two numbers of opposite qualities, *find the difference of their arithmetical values and prefix to it the sign of the arithmetically greater number.*

For example, to add + 9 and – 5, we write + 9 as the sum of + 4 and + 5, and obtain

$$\begin{aligned}(+9)+(-5) &= (+4)+(+5)+(-5)\\ &= +4.\end{aligned}$$

Thus if a man's assets amount to \$9, and he owes \$5, he is worth \$4; that is, (+ \$9) + (– \$5) = + \$4.

Again, to add – 11 and + 4, we have

$$\begin{aligned}(-11)+(+4) &= (-7)+(-4)+(+4)\\ &= -7.\end{aligned}$$

Thus if a man owes \$11, and his assets amount to \$4, he is in debt \$7; that is,

$$(-\$11)+(+\$4) = -\$7.$$

(iii.) To add three or more numbers of opposite qualities; *by* (i.) *find the sum of all the positive numbers, and then of all the negative numbers. Then by* (ii.) *add these sums.*

For example, to add -5, $+9$, -11, and $+6$, we have

$$\begin{aligned}&(-5)+(+9)+(-11)+(+6)\\ &\quad=(+9)+(+6)+(-5)+(-11) &&\text{by (A)}\\ &\quad=(+15)+(-16) &&\text{by (B)}\\ &\quad=-1.\end{aligned}$$

Adding the numbers in the order given, we have

$$\begin{aligned}&(-5)+(+9)+(-11)+(+6)\\ &\quad=(+4)+(-11)+(+6)\\ &\quad=(-7)+(+6)=-1.\end{aligned}$$

Example. Add $\frac{1}{3}$, $-\frac{1}{2}$, $\frac{1}{4}$, and $-\frac{1}{6}$.

$$\begin{aligned}&(+\tfrac{1}{3})+(-\tfrac{1}{2})+(+\tfrac{1}{4})+(-\tfrac{1}{6})\\ &\quad=+(\tfrac{4}{12})+(+\tfrac{3}{12})+(-\tfrac{6}{12})+(-\tfrac{2}{12})\\ &\quad=(+\tfrac{7}{12})+(-\tfrac{8}{12})=-\tfrac{1}{12}.\end{aligned}$$

43. Rule for Subtraction. To subtract one algebraic number from another,

Add to the minuend the subtrahend with its quality changed.

For let s denote the subtrahend, and r the remainder; then, by § 17, $r+s$ will denote the minuend. Let $-s$ denote the subtrahend with its quality changed; then

$$r+s+(-s)=r;$$

that is, the minuend, $r+s$, plus the subtrahend with its quality changed, $-s$, equals the remainder; hence the rule.

Thus let $+5=$ the subtrahend,
and $+11=$ the remainder;
then $+11+(+5)=$ the minuend.
Now $-5=$ the subtrahend with quality changed,
and $+11+(+5)+(-5)=+11$, the remainder.

Again, if a man is worth \$700, subtracting \$200 of credit is equivalent to adding \$200 of debt, and subtracting \$300 of debt is equivalent to adding \$300 of credit; that is,

$$\$700-(+\$200)=\$700+(-\$200)=\$500,$$
and
$$\$700-(-\$300)=\$700+(+\$300)=\$1000.$$

Example 1. From $+9$ take $+4$.

$$9-(+4)=9+(-4)=+5.$$

Since $9+(-4)=9-(+4)$, $9-4$ may be regarded as denoting either $9+(-4)$ or $9-(+4)$.

Example 2. From 3 take -15.

$$3-(-15)=3+(+15)=+18.$$

Hence subtracting a number is equivalent to adding the same number with its quality changed.

44. Laws of Subtraction. Since any problem of algebraic subtraction can be reduced to a problem of addition, both the law of order and the law of grouping hold for subtraction.

Exercise 6.

Find the sum of

1. $5, -7, 4, -2.$
2. $4, -1, 6, -8.$
3. $-9, 5, -7, 6, -15, 11.$
4. $-10, 9, -8, 14, -6, 4.$
5. $\frac{1}{3}, -\frac{1}{4}, -\frac{7}{8}, \frac{1}{2}.$
6. $\frac{5}{6}, -\frac{11}{12}, \frac{3}{4}, -\frac{3}{8}.$
7. $-\frac{13}{14}, \frac{2}{7}, -\frac{15}{28}, \frac{3}{4}.$
8. $\frac{7}{15}, -\frac{2}{3}, \frac{31}{45}, -\frac{4}{5}.$
9. From 7 take 4.
10. From 4 take 7.
11. From 2 take -9.
12. From -3 take 4.
13. From -8 take -9.
14. From -15 take -7.
15. From $4+(-8)+9+(-3)$ take $7+(-2)+9+(-8)$.
16. From $(-10)+(-7)+15+(-3)$ take $7+(-11)+(-17)$.

17. From $12+4+16+(-30)$ take $8+(-9)+(-11)+13$.

18. From $12-9-7+2$ take $-8+5+7$.

19. From $-12+7-15+9$ take $-9+8-7+6$.

20. From $-13-15+20$ take $-7-6+4-2$.

21. From $\frac{2}{3}-\frac{1}{2}+\frac{1}{4}-\frac{7}{8}$ take $\frac{3}{4}-\frac{5}{8}-\frac{1}{2}-\frac{2}{3}$.

22. From $-\frac{2}{5}-\frac{2}{3}+\frac{11}{15}+\frac{13}{45}$ take $-\frac{14}{15}-\frac{3}{5}-\frac{38}{45}+\frac{1}{3}$.

MULTIPLICATION.

45. Powers. If the factors of a product are all equal, the product is called a **Power** of that factor. Any number also is often called the first power of itself.

Thus aa is called the *second power* of a; aaa, the *third power* of a; and $aaa \cdots$ to n factors, the n*th power* of a.

46. An **Exponent** is a number whose symbol is placed to the right and a little above that of another number, which is called the **Base.**

A **Positive Integral** exponent denotes how many times the base is to be taken as a factor.

Thus a^2, read 'a square,' denotes $a \cdot a$;
a^3, read 'a cube,' denotes $a \cdot a \cdot a$;
and a^n, read 'a nth power,' denotes $a \cdot a \cdot a \cdots$ to n factors.

Here a is the base, and 2, 3, and n are exponents.

The meaning of fractional and negative exponents will be found in Chapter XIX.

47. Equimultiples of two or more expressions are the products obtained by multiplying each of them by the same expression.

Thus Am and Bm are equimultiples of A and B.

48. *Two expressions are identical if their equimultiples are identical.*

Let A and B denote any two expressions, such that

$$Am \equiv Bm; \qquad (1)$$

then

$$A \equiv B. \qquad (2)$$

For, dividing each member of (1) by m, by Axiom 4 of § 28 we obtain identity (2).

49. Law of Quality. *Two like signs give* $+$; *two unlike signs give* $-$.

If, in the definition of a product in § 19, we take 1 to denote $+1$, we have

$$+3=0+(+1)+(+1)+(+1).$$

$$\therefore\ (+4)\times(+3)=+(+4)+(+4)+(+4)=+12, \qquad (1)$$

and

$$(-4)\times(+3)=+(-4)+(-4)+(-4)=-12. \qquad (2)$$

Again,

$$-3=0-(+1)-(+1)-(+1).$$

$$\therefore\ (+4)\times(-3)=-(+4)-(+4)-(+4)=-12, \qquad (3)$$

and

$$(-4)\times(-3)=-(-4)-(-4)-(-4)=+12. \qquad (4)$$

From (1) and (4) it follows that *two factors like in quality give a positive product;* and from (2) and (3) it follows that *two factors opposite in quality give a negative product.*

Thus the product ab is positive or negative according as a and b are like or unlike in quality.

Example 1. Multiply $+7$ by -5.

By the law of quality the product is $-$, and by arithmetic its absolute value is 35; hence

$$(+7)\times(-5)=-35.$$

Example 2. Find the product of -6, $+4$, and -3.

$$\begin{aligned}(-6)\times(+4)\times(-3)&=(-24)\times(-3)\\&=+72.\end{aligned}$$

50. *Any product containing an odd number of negative factors will be negative, all others will be positive.*

For
$$(+1)(-1)(-1)=(-1)(-1)=+1;$$
$$(-1)(+1)(-1)(-1)=(-1)(-1)(-1)$$
$$=(+1)(-1)=-1;$$
$$(-1)(-1)(+1)(-1)(-1)(-1)$$
$$=(+1)(+1)(-1)(-1)(-1)=-1;$$

and so on, whatever be the *order* or the *grouping* of the factors.

Thus the term $5\,abcd$ is negative if an odd number of its factors are negative, otherwise it is positive.

51. *The quality of any term is changed by changing the quality of one of its factors.*

For, by changing the quality of one factor, the number of negative factors is changed from odd to even, or from even to odd; hence, by § 50, the quality of the term is changed.

Thus the terms $-4\,abc$ and $4\,abc$ are opposite in quality; so also are $(a+b)\,x$ and $(a+b)\,(-x)$.

52. The quality of any expression is changed by changing the quality of each of its terms.

Thus the expressions $2\,ax-3\,b+4\,cx^2$ and $-2\,ax+3\,b-4\,cx^2$ are opposite in quality, but they have the same arithmetical value.

This principle is illustrated by the fact that if in a business account we change debts into credits and credits into debts, the balance will not be changed in amount, but it will be changed in quality.

53. Laws of Order and Grouping. By § 50, any change in the *order* or *grouping* of factors will not affect the *quality* of the product; and, by §§ 20 and 21, this will not affect its arithmetical value. Hence both the law of order and the law of grouping of factors hold for algebraic multiplication.

54. Distributive Law. *The product of two expressions is equal to the sum of the products obtained by multiplying each term of either expression by the other;* that is,

$$(a+b+c+\cdots)\,x \equiv ax+bx+cx+\cdots. \qquad \text{(C)}$$

Let m and n denote any *positive integers,* and a and b any numbers whatever; then, by § 53, we have

$$\begin{aligned} (a+b)m &\equiv (a+b)+(a+b)+\cdots \text{ to } m \text{ terms} \\ &\equiv am+bm. \qquad (1) \end{aligned}$$

Again, $(a+b)(m\div n) \equiv a(m\div n)+b(m\div n). \qquad (2)$

For $(a+b)(m\div n)n$

$$\equiv (a+b)m \equiv am+bm; \qquad \text{by (1)}$$

and $[a(m\div n)+b(m\div n)]n$

$$\begin{aligned} &\equiv a(m\div n)n+b(m\div n)n \qquad \text{by (1)} \\ &\equiv am+bm. \qquad \text{§ 23} \end{aligned}$$

That is, equimultiples of the members of (2) are identical; hence, by § 48, the members of (2) are identical.

The same reasoning would apply to any polynomial as well as to $a+b$; hence (C) is proved for positive values of x.

Now changing the quality of the members of (C), we obtain the identity,

$$(a+b+c+\cdots)(-x) \equiv a(-x)+b(-x)+c(-x)+\cdots,$$

which proves the law for negative multipliers.

Thus $(a+b)\times \frac{5}{3} \equiv a\times\frac{5}{3}+b\times\frac{5}{3}$;

and $(a+b)(-\frac{5}{3}) \equiv a(-\frac{5}{3})+b(-\frac{5}{3})$.

55. Law of Exponents. *The product of the* m*th and the* n*th powers of any base equals the* (m + n)*th power of that base.*

For, if m and n denote any positive integers, by § 46 we have

$$\begin{aligned} a^m \cdot a^n &= (aaa \cdots \text{to } m \text{ factors})(aaa \cdots \text{to } n \text{ factors}) \\ &= aaa \cdots \text{to } m + n \text{ factors} && \text{by (B}'\text{)} \\ &= a^{m+n}. && \text{§ 46} \end{aligned}$$

Thus $a^3a^2 = aaa \cdot aa = a^{3+2} = a^5.$

Example. Multiply $3a^2x^3$ by $-4a^4x^2y$.

$$\begin{aligned} (3a^2x^3)(-4a^4x^2y) &= 3a^2x^3(-4)a^4x^2y && \text{by (B}'\text{)} \\ &= 3(-4)a^2a^4x^3x^2y && \text{by (A}'\text{)} \\ &= -12a^6x^5y && \text{by (B}'\text{), § 55} \end{aligned}$$

$$\begin{array}{r} 3a^2x^3 \\ -\ \ 4a^4x^2y \\ \hline -12a^6x^5y \end{array}$$

The operation may be performed by writing the coefficients and like letters in vertical columns, as in the margin, and then multiplying together the factors in these columns. Or we may change the grouping and the order of the factors and obtain the product mentally.

Exercise 7.

Find the product of

1. $+2$ and $+7$.
2. -4 and $+17$.
3. -7 and $+27$.
4. -9 and $+28$.
5. $+5, -7, +4, -2, -1$.
6. $-8, -1, -2, -3, -4$.
7. $+2, -1, -1, -1, +9$.
8. $+7, -2, -2, -1, -1$.

If $a = +2$, $b = -4$, $c = -5$, $d = -3$, $x = +6$, find the value of

9. $5a + bx$.
10. $ax - bc$.
11. $4b - abx$.
12. $b^2 + c^2$.
13. $c^2 - x^2$.
14. $b^3 + d^3$.
15. $b^2(x + c)$.
16. $2ab(c - d^2)$.
17. $bc(x^2 - c^3)$.

If $a = -5$, $b = +3$, $c = -4$, $x = -2$, $y = +1$, find the value of

18. $a^3 + 2b^3 - 3c^3 + x^3$.
19. $5abc - a^2cx + 4cy$.
20. $acy - x^3 + acy$.
21. $a^2 + 2b^3 - 3c^4$.
22. $c^3 + ab^2x^3 + y$.
23. $3abcx - 4a^2c^2 - a^3x^3$.
24. $a^2cx - 4aby + 3abcx$.
25. $7abcxy - 4cx^2y^4 - 3a^2$.

Find the product of

26. a^3 and a^4.
27. a^7 and a^5.
28. y^2, y^3, and y^4.
29. ax and $3ax$.
30. $-2abx$ and $-7ab$.
31. $6x^2y$ and $-10axy$.
32. $-3a^2b$ and $12ab^3$.
33. $-abcd$ and $-3a^2b^2c$.
34. $7x^2y^2z^3$ and $-5x^2y^3z$.
35. $-3a^2b^3c^2$ and $8a^3b^5c^4d$.
36. $2ab$, $-4a^2b$, and $5ab^3$.
37. $-5ax$, $-7a^2x$, and $2ax^3$.
38. $8xy^3$, $-3x^2y$, and $-3xy$.
39. $-7ab^2$, $-3a^2b^3$, and $-a^2b^2$.
40. a^2b^2c, $2ab^3c$, and $-5abc$.
41. $-7x^2y^3$, x^3y^4, and axy.
42. $-a^2bx$, ab^2x, and $-ax^5$.
43. $-a^2x^3$, $-b^2x$, and $-aby$.

44. Show that the quality of a term is changed by changing the quality of an odd number of its factors.

45. Show that the quality of a term is not changed by changing the quality of an even number of its factors.

CHAPTER IV.

ADDITION AND SUBTRACTION.

56. Laws of Addition and Subtraction.

Law of Order. Numbers to be added may be arranged in any order; that is,

$$a + b = b + a. \qquad \text{(A)}$$

Law of Grouping. Numbers to be added may be grouped in any manner; that is,

$$a + b + c = a + (b + c). \qquad \text{(B)}$$

Distributive Law. Expressed in symbols this law is,

$$ax + bx + cx + \cdots = (a + b + c + \cdots)x. \qquad \text{(C)}$$

57. Addition of Monomials. By law (C), *similar terms may be united into one term by adding their coefficients and affixing to the result the common letters with their exponents.*

EXAMPLE. Add $3\,ab^2$, $-5\,ab^2$, and $-8\,ab^2$.

$$\begin{aligned}(+3)\,ab^2 + (-5)\,ab^2 + (-8)\,ab^2 &= (3 - 5 - 8)\,ab^2\\ &= -10\,ab^2.\end{aligned}$$

To add dissimilar terms,

Write them in succession with their signs unchanged.

For any algebraic expression denotes the sum of all its terms, the sign before each term denoting the quality of its numerical coefficient.

Thus the sum of $-5\,c$, $7\,a$, and $-9\,b$ is $-5\,c + 7\,a - 9\,b$, or $7\,a - 5\,c - 9\,b$, the order of the terms being changed by the law of order.

Exercise 8.

Find the sum of

1. $4ab^2, 7ab^2, 3ab^2.$
2. $-3x, -5x, -4x.$
3. $-2c^2, 5c^2, -8c^2.$
4. $-5ac, ac, 3ac.$
5. $4by^2, -7by^2, 9by^2.$
6. $5ax^2, -3bx^2, -8cx^2.$

$$(+5a)x^2 + (-3b)x^2 + (-8c)x^2 = (5a - 3b - 8c)x^2.$$

7. $7ax^2, -5ax^2, 4ax^2, -9ax^2, -14ax^2, 25ax^2.$
8. $9ax^2, -ax^2, 4ax^2, -7bx^2, -14cx^2.$

Simplify the following expressions by combining like terms:

9. $-9x^2 + 17x^2 + 3x - 4x^2 - x^2 + y^2 - 5y^2.$
10. $3ab^2 - 7ab^2 + 8ab^2 - 4ab^2 + 7cx^2 - 11cx^2.$
11. $-12x^3 + 4x^3 - 9x^3 + 7x^3 + 8a^3 - 9a^3 + 7a^3.$
12. $7abcd - 11\,abcd + 41abcd + 7xy - 20xy.$
13. $-\frac{5}{2}x^2 - 2x^2 + \frac{5}{8}x^2 + 8y^2 - \frac{3}{4}y^2 - \frac{5}{8}y^2.$
14. $7x^2 + 2a^2 - 5x^2 - 3a^2.$

$$\begin{aligned} 7x^2 + 2a^2 - 5x^2 - 3a^2 &= 7x^2 - 5x^2 + 2a^2 - 3a^2 \qquad \text{by (A)} \\ &= 2x^2 - a^2. \end{aligned}$$

15. $7ab - 5xy + 3ab + 2xy - 6ab + xy.$
16. $-9ax^2 + 5by^2 + 7ax^2 - 3by^2 + 11ax^2 + 4by^2.$
17. $-7cy^2 - 4ab + 9xz + 11cy^2 + 10ab - 5xz - ab.$

58. Addition of Polynomials. To add two or more algebraic expressions,

Write down all the terms in succession with their signs unchanged, and then unite the similar terms if any.

For by the law of grouping a polynomial may be added to any expression by adding each of its terms.

Example 1. Add $5x^2+7x$ and $3x^2-4x$

$$\begin{aligned}5x^2+7x+(3x^2-4x)&=5x^2+7x+3x^2-4x && \text{by (B)}\\ &=5x^2+3x^2+7x-4x && \text{by (A)}\\ &=8x^2+3x. && \text{by (C)}\end{aligned}$$

The operation is more conveniently performed by writing the similar terms in vertical columns and then adding the terms in these columns as at the right.

$$\begin{array}{r}5x^2+7x\\ 3x^2-4x\\ \hline 8x^2+3x\end{array}$$

Example 2. Add $4x^2-3xy+y^2$, $2x^2-5xy-6y^2$, and $2xy-x^2-3b^2$.

Writing the like terms in vertical columns, we have

$$\begin{array}{rrrr}4x^2 & -3xy & +y^2 & \\ 2x^2 & -5xy & -6y^2 & \\ -x^2 & +2xy & & -3b^2\\ \hline 5x^2 & -6xy & -5y^2 & -3b^2\end{array}$$

59. A polynomial is said to be *arranged according to the powers* of some letter when the exponents of that letter either decrease or increase in the successive terms.

Thus ax^3+bx^2-cx+d is arranged according to the descending powers of x, and $4-7x+ax^2-cx^3$ is arranged according to the ascending powers of x. In adding polynomials, it is often convenient to arrange them according to the powers of some letter, as in the examples of § 58.

Exercise 9.

Add

1. $a+2b-3c,\ -3a+b+2c,\ 2a-3b+c.$
2. $-3x+2y+z,\ x-3y+2z,\ 2x+y-3z.$
3. $-15a-19b-18c,\ 14a+15b+8c,\ a+5b+9c.$
4. $5ax-7by+cz,\ ax+2by-cz,\ -3ax+2by+3cz.$
5. $20p+q-r,\ p-20q+r,\ p+q-20r.$
6. $-5ab+6bc-7ac,\ 8ab-4bc+3ac,\ -2ab-2bc+4ac.$

7. $pq + qr - pr,\ -pq + qr + pr,\ pq - qr + pr.$

8. $2ab + 3ac + 6abc,\ -5ab + 2bc - 5abc,\ 3ab - 2bc - 3ac.$

9. $x^2 + xy - y^2,\ -z + yz + y^2,\ -x^2 + xz + z^2.$

10. $5a^3 - 3c^3 + d^3,\ b^3 - 2a^3 + 3d^3,\ 4c^3 - 2a^3 - 3d^3.$

11. $x^2 + y^2 - 2xy,\ 2z^2 - 3y^2 - 4yz,\ 2x^2 - 2z^2 - 3xz.$

12. $x^3 + 3x^2y + 3xy^2,\ -3x^2y - 6xy^2 - x^3,\ 3x^2y + 4xy^2.$

13. $x^5 - 4x^4y - 5x^3y^3,\ 3x^4y + 2x^3y^3 - 6xy^4,\ 3x^3y^3 + 6xy^4 - y^5.$

14. $a^3 - 4a^2b + 6abc,\ a^2b - 10abc + c^3,\ b^3 + 3a^2b + abc.$

15. $3a^2 - 10b^2 + 5c^2 - 7bc,\ -a^2 + 4b^2 - 10c^2 + 3ab,\ c^2 + 11bc + 8ac - 2ab,\ 4c^2 - 4bc + ac,\ -2a^2 + 6b^2 - 9ac - bc.$

16. $4x^5 + 12x^3 - x - 10,\ 11x^2 - 2x^4 - x^5 + 9,\ 9x^2 - 3x^5 + 4x,\ 4x^3 - x^4 - 5,\ 6x^4 - x^3 + 2x^2 - 7.$

17. $\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{4},\ -\frac{3}{4}x^2 + \frac{1}{4}x - \frac{3}{4},\ \frac{5}{8}x^2 + \frac{2}{3}x + \frac{1}{2}.$

18. $\frac{2}{3}a^2 + \frac{1}{3}ab - \frac{1}{4}b^2,\ \frac{2}{3}a^2 - ab - \frac{5}{4}b^2,\ -a^2 - \frac{2}{3}ab + 2b^2.$

19. $-\frac{2}{3}x^2 - xy + y^2,\ 3x^2 - \frac{2}{5}xy - \frac{1}{2}y^2,\ -\frac{3}{2}x^2 + 2xy - \frac{2}{3}y^2.$

20. $-\frac{3}{2}x^3 - \frac{3}{4}xy^2 + 2y^3,\ \frac{3}{2}x^2y + xy^2 + \frac{1}{2}y^3,\ \frac{1}{2}x^3 - 2x^2y - \frac{3}{2}y^3.$

60. Rule for Subtraction. By § 52, the quality of an expression is changed by changing the sign before each of its terms; hence, by § 43,

To subtract one expression from another,

Change the sign before each term of the subtrahend and add the result to the minuend.

EXAMPLE 1. From $-5x^2y$ take $4x^2y$.

$$\begin{aligned} -5x^2y - (+4x^2y) &= -5x^2y + (-4x^2y) \\ &= -9x^2y. \end{aligned}$$

Example 2. From $5x^2 + xy - m$ take $2x^2 + 8xy - 7y^2$.

Changing the sign before each term of the subtrahend and adding the result to the minuend, we have

$$\begin{array}{rrrr}
5x^2 + & xy - & m & \\
-2x^2 - & 8xy & & + 7y^2 \\
\hline
3x^2 - & 7xy - & m & + 7y^2,
\end{array} \text{ Remainder.}$$

Note. The signs of the subtrahend need not be *actually* changed; the operation of changing the signs ought usually to be performed *mentally*, as in the following example.

Example 3. From $2x^4 - 3x^2 + 7x - 8$ take $x^4 - 2x^3 - 9x + 4$.

$$\begin{array}{rrrrr}
2x^4 & & -3x^2 + & 7x - & 8 \\
x^4 - & 2x^3 & - & 9x + & 4 \\
\hline
x^4 + & 2x^3 - & 3x^2 + & 16x - & 12
\end{array}$$

Exercise 10.

1. From $4a - 3b + c$ subtract $2a - 3b - c$.
2. From $15x + 10y - 18z$ subtract $2x - 8y + z$.
3. From $-10bc + ab - 4cd$ take $-11ab + 6cd$.
4. From $ab + cd - ac - bd$ take $ab + cd + ac + bd$.
5. From $m^2 + 3n^3$ subtract $-4m^2 - 6n^3 + 71x$.
6. $7xy - (-3xy) = ?$
7. $-9x^2y - (+6x^2y) - (-20x^2y) = ?$
8. $32x^2 - (-12y^2) - (+14x^2) - (+9y^2) + (-2y^2) = ?$
9. $28a^2b^2 - (+17a^2b^2) - (-19x^2y) - (+15x^2y)$
$-(-5a^2b^2) = ?$

From

10. $-8x^2y^2 + 15x^3y + 13xy^3$ take $4x^2y^2 + 7x^3y - 8xy^3$.
11. $a^2bc + b^2ca + c^2ab$ take $3a^2bc - 5b^2ca - 4c^2ab$.
12. $-7a^2b + 8ab^2 + cd$ take $5a^2b - 7ab^2 + 6cd$.

13. $10a^2b^2 + 15ab^2 + 8a^2b$ take $-10a^2b^2 + 15ab^2 - 8a^2b$.

14. $b^3 + c^3 - 2abc$ take $a^3 + b^3 - 3abc$.

15. $7abc - 3a^3 + 5b^3 - c^3$ take $a^3 + b^3 + c^3 - 3abc$.

16. $\frac{1}{2}x^2 - \frac{1}{3}xy - \frac{3}{2}y^2$ take $-\frac{3}{2}x^2 + xy - y^2$.

17. $\frac{3}{8}x^2 - \frac{2}{3}ax$ take $\frac{1}{3} - \frac{1}{4}x^2 - \frac{5}{6}ax$.

18. $\frac{1}{3}a^3 - 2ax^2 - \frac{1}{3}a^2x$ take $\frac{1}{3}a^2x + \frac{1}{4}a^3 - \frac{3}{2}ax^2$.

If $A = a^2 - 4ab - 3b^2$, $\qquad B = ab - 4b^2 - 3a^2$,

$C = b^2 - 4a^2 - 3ab$, $\qquad D = 2a^2 + 2b^2 + 2c^2$,

find the expression for

19. $A + B + C + D$.

20. $A + B + C - D$.

21. $A + B - C - D$.

22. $A - B - C - D$.

23. $-A - B + C + D$.

24. $-A + B - C + D$.

In solving example 20, under the values of A, B, and C write that of D with its quality changed, and then add the results.

25. From $5x^3 + 3x - 1$ take the sum of $2x - 5 + 7x^2$ and $3x^2 + 4 - 2x^3 + x$.

26. From the sum of $2a^3 - 3a^2 + a - 2$ and $2 + 8a^2 - a^3$ subtract $3a - 7a^3 + 5a^2$.

27. From the sum of $4x^3 + 3x - 7$, $2x^2 - 3x + 2x^3 - 1$, and $-5x^3 + 2x - x^2 + 9$ take the sum of $2x^3 - 11x$ and $9x^3 + 5x^2 + 3 - 2x$.

61. As an *arithmetical* expression, $6 - 5$ denotes simply that 5 is to be subtracted from 6, the sign $-$ being a sign of operation. The expression $a - b$ is meaningless in arithmetic when $b > a$.

As an *algebraic* expression, $6 - 5$ denotes either the sum $(+6) + (-5)$, or the difference $(+6) - (+5)$. In either case the sign $+$ is understood, in the first case as a sign of operation, and in the second case as a sign of quality. Since $(+6)+(-5)\equiv(+6)-(+5)$, the expression $6-5$ denotes the same number whichever view we take. As stated in § 40, we shall *usually* regard a polynomial as the sum of its terms, the sign before each term denoting the quality of its numerical coefficient.

62. Removal of Signs of Grouping. To indicate that any polynomial is to be added or subtracted, we enclose it by a sign of grouping, and prefix the sign + for addition, and the sign − for subtraction. Hence, by §§ 58 and 60,

A sign of grouping preceded by the sign + may be removed without changing the sign before any enclosed term.

A sign of grouping preceded by the sign − may be removed, if the sign before each enclosed term be changed from + to − or from − to +.

Thus
$$\begin{aligned}3a+(4a-5b+m)&=3a+4a-5b+m\\&=7a-5b+m,\end{aligned}$$
and
$$\begin{aligned}5a-(3b-2a+4c)&=5a-3b+2a-4c\\&=7a-3b-4c.\end{aligned}$$

When one sign of grouping is enclosed within another, to prevent confusion, different forms of these signs should be used. The beginner should remove only one sign of grouping at a time, beginning with the *innermost* one.

EXAMPLE. Remove the signs of grouping from the expression $a-[x+(y-\overline{b-c})-z]$.

$$\begin{aligned}a-[x+(y-\overline{b-c})-z]&=a-[x+(y-b+c)-z]\\&=a-[x+y-b+c-z]\\&=a-x-y+b-c+z.\end{aligned}$$

In the above process the vinculum was removed first, then the parentheses ( ), and then the brackets [ ]. The sign $-$ before $\overline{b-c}$ affects the whole binomial, the coefficient of b being $+1$.

Removing the outer sign of grouping first, we have

$$\begin{aligned} a-[x+(y-\overline{b-c})-z] &= a-x-(y-\overline{b-c})+z \\ &= a-x-y+\overline{b-c}+z \\ &= a-x-y+b-c+z. \end{aligned}$$

The more advanced student should begin with the outer sign of grouping, as he can thereby soon learn to remove, without error, two or more signs of grouping at a time.

Exercise 11.

Simplify each of the following expressions by removing the signs of grouping and collecting like terms:

1. $a-(b+c)+(b-c-a)$.
2. $3x-(y-2x)+(z+y-5x)$.
3. $z-\{y-(z-x)\}$.
4. $3x-\{2y+5z-(3x+y)\}$.
5. $a-[a-\{a-(2a-a)\}]$.
6. $1-[2-\{3-(4-5)\}]$.
7. $a+b-[a-b+\{a+b-(a-b)\}]$.
8. $x-(y-z)+\{2z-\overline{3y-5x}\}$.
9. $5-[4+\{5-(4+\overline{5-4})\}]$.
10. $2a-\{3b+(4c-\overline{3b+2a})\}$.
11. $a-2b-\{3a-(b-c)-5c\}$.
12. $a-[3b+\{3c-(d-b)+a\}-2a]$.
13. $2x-(5y-\overline{3z+7})-[4+\{x-(3y+2z+5)\}]$.
14. $3a-[2b-\{4c-12a-(4b-8c)\}-(6b-12c)]$.
15. $-[15x-\{14y-(15z+12y)-(10x-15z)\}]$.

16. $-[a-\{a+(x-a)-(x-a)-a\}-2a]$.

17. $2x-(3y-4z)-\{2x-(3y+4z)\}$
$-\{3y-(4z+2x)\}$.

63. Insertion of Signs of Grouping. From § 62 we have,

Any number of terms of a polynomial may be enclosed in a sign of grouping preceded by the sign + without changing the sign before any enclosed term.

Any number of terms of a polynomial may be enclosed in a sign of grouping preceded by the sign —, if the sign before each enclosed term be changed from + to — or from — to +.

Thus $5x-7y+4c-7b=5x+(-7y+4c-7b)$;

$$7x-6y+5a-6c=7x-(6y-5a+6c);$$

$$ax^3-2cx-cx^3+bx^2-x=(ax^3-cx^3)+bx^2-(2cx+x);$$

$$=(a-c)x^3+bx^2-(2c+1)x.$$

Exercise 12.

In each of the following expressions enclose the last four terms in a sign of grouping preceded by the sign —, without changing the value of the expression:

1. $3x-2a+5b-y+z$.

2. $a-b-x+3b-z+2y$.

3. $3y+2x+7z+a+2b+c$.

4. $2z-7x-2a-3b-5c-9y$.

Simplify each of the following expressions by combining the terms having the same powers of x, so as to have the sign + before each sign of grouping:

5. $ax^4+bx^2+5+2bx-5x^2+2x^4-3x$.

Ans. $(a+2)x^4+(b-5)x^2+(2b-3)x+5$.

6. $3bx^2 - 7 - 2x + ab + 5ax^3 + cx - 4x^2 - bx^3.$

7. $2 - 7x^3 + 5ax^2 - 2cx + 9ax^3 + 7x - 3x^2.$

8. $2cx^5 - 3abx + 4dx - 3bx^4 - a^2x^5 + x^4.$

Simplify each of the following expressions by combining like terms in x so as to have the sign $-$ before each sign of grouping:

9. $ax^2 + 5x^3 - a^2x^4 - 2bx^3 - 3x^2 - bx^4.$

10. $7x^3 - 3c^2x - abx^5 + 5ax + 7x^5 - abcx^3.$

11. $ax^2 + a^2x^3 - bx^2 - 5x^2 - cx^3.$

12. $3b^2x^4 - bx - ax^4 - cx^4 - 5c^2x - 7x^4.$

Simplify the following expressions, and in each result add the terms involving like powers of x:

13. $ax^3 - 2cx - [bx^2 - \{cx - dx - (bx^3 + 3cx^2)\}$
$- (cx^2 - bx)].$

14. $5ax^3 - (7bx - 7cx^2) - \{6bx^2 - (3ax^2 + 2ax) - 4cx^3\}.$

15. Add $a^2x^3 - 5x$, $2ax^2 - 5ax^3$, $2x^3 - bx^2 - ax.$

16. Add $ax^2 + bx - c$, $qx - r - px^2$, $x^2 + 2x + 3.$

17. Add $px^3 - qx$, $qx^2 - px$, $q - x^3$, $px^2 + qx^3.$

CHAPTER V.

MULTIPLICATION.

64. Laws of Multiplication.

Law of Order. Factors may be arranged in any order; that is,

$$ab = ba. \tag{A$'$}$$

Law of Grouping. Factors may be grouped in any manner; that is,

$$abc = a(bc). \tag{B$'$}$$

Distributive Law. Stated in symbols this law is

$$(a + b + c + \cdots)x = ax + bx + cx + \cdots. \tag{C}$$

Law of Quality. $\left\{\begin{array}{l}\text{Two like signs give } +;\\ \text{Two unlike signs give } -.\end{array}\right\}$ (D)

Law of Exponents. $a^m a^n = a^{m+n}$. (E)

65. For the **multiplication of monomials,** by the laws given above we have the following rule:

Multiply together their numerical coefficients, observing the law of quality; after this write the product of their literal factors, observing the law of exponents.

EXAMPLE. Multiply together $-5\,ay^2$, $-2\,a^2x^3$, and $-9\,ax^2y$.

$$\begin{aligned}(-5\,ay^2)(-2\,a^2x^3)(-9\,ax^2y) &= (-5)(-2)(-9)\,aa^2ax^3x^2y^2y\\ &= -90\,a^4x^5y^3.\end{aligned}$$

66. For the **multiplication of a polynomial by a monomial,** by the *distributive* law we have the following rule:

Multiply each term of the polynomial by the monomial, and add the partial products.

Example 1. Multiply $2z^2 - 4y^3$ by $-3yz$.

$$\begin{aligned}(2z^2 - 4y^3)(-3yz) &= (2z^2)(-3yz) + (-4y^3)(-3yz)\\ &= -6yz^3 + 12y^4z.\end{aligned}$$

Or, writing the multiplier under the multiplicand, the work may be arranged as at the right of the page.

$$\begin{array}{r} 2z^2 - 4y^3 \\ -3yz \\ \hline -6yz^3 + 12y^4z \end{array}$$

Exercise 13.

Write out the product of

1. $-2abx$ and $-7ab^2$.
2. $6x^2y$ and $-10axy$.
3. $-3a^2b$ and $12ab^3$
4. $-\frac{1}{6}ab^2cd^2$ and $-3a^2b^2c$.
5. $\frac{1}{2}x^2y^2z^3$ and $-\frac{1}{5}x^2y^3z$.
6. $-\frac{1}{7}a^2b$ and $7ab^2c^3$.
7. $-2ab$, $-4a$, and $-5b^2$.
8. $-\frac{1}{3}a^2x$, $\frac{1}{2}ax^2$, and $-3a^2x$.
9. $-\frac{1}{2}a^2b^2c$, $-7abc^2$, and $-\frac{1}{4}b^3c$.
10. $-\frac{1}{2}ax$, $-\frac{1}{3}xy$, and $-\frac{2}{5}ay$.
11. $-\frac{3}{4}x^2y$, $-\frac{5}{6}xy^3$, and $-\frac{1}{5}y$.
12. $-0.1xyz$ and $0.01x^2yz^2$.

Multiply

13. $4a^2 - 5a + 3b$ by $2a^3$.
14. $2a^2 + 3ab + 2b^2$ by $-3a^2b^2$.
15. $bc + ca - ab$ by abc.
16. $2x^3 - 3x^2 + 5x - 4$ by $-5x^2$.
17. $-4x^4 + 3x^3 - 3x^2 + 4$ by $-6x^3$.
18. $9gh - 12ga - 3gb$ by $3gh$.

19. $-a^2bc+b^2ca-c^2ab$ by $-ab$.

20. $-5xy^2z+3xyz^2-8x^2yz-7xyz$ by $-2xyz$.

21. $a^2b^2c^2-abc-ax-by-cz$ by $-5abcxy$.

22. $\frac{6}{7}a^2x^2-\frac{3}{2}ax^3+\frac{1}{5}ax$ by $-\frac{7}{3}a^3x$.

23. $-\frac{3}{2}xy^2+\frac{1}{2}axy-\frac{2}{5}ay^2+\frac{4}{3}a^2x$ by $-\frac{2}{3}axy$.

24. $\frac{7}{11}x^2y^3-\frac{2}{3}xy^2+\frac{1}{2}axy-\frac{1}{5}x^3y^2$ by $\frac{11}{5}x^2y$.

By removing the signs of grouping, simplify each of the following expressions:

25. $(a+b)c-(a-b)c$. 27. $\frac{1}{2}(b-2c)+\frac{3}{4}(c-2b)$.

26. $2(a-b)+4(a+b)$. 28. $7a(b-c)-2b(a-c)$.

29. $a^2b^2(c^2-d^2)+c^2d^2(a^2-b^2)+b^2c^2(d^2-a^2)$.

30. $2\{3ab-4a(c-2b)\}$.

31. $7ac-2\{2c(a-3b)-3(5c-2b)a\}$.

67. Polynomials by Polynomials. Let X denote the multiplicand, and $a+b+c+\cdots$ the multiplier; then, by laws (C) and (A′) we have

$$X(a+b+c+\cdots)=Xa+Xb+Xc+\cdots.$$

Hence to multiply one polynomial by another,

Multiply the multiplicand by each term of the multiplier and add the several products.

EXAMPLE 1. Multiply $2x-3y$ by $4x-7y$.

Write the multiplier under the multiplicand, and draw a line under them as below.

$$\begin{array}{lll}
2x & -\ 3y & \\
4x & -\ 7y & \\
\hline
8x^2 & -\ 12xy & \\
 & -\ 14xy & +\ 21y^2 \\
\hline
8x^2 & -\ 26xy & +\ 21y^2
\end{array}$$

Multiplying the multiplicand $2x - 3y$ by $4x$, we obtain the product $8x^2 - 12xy$, which we write under the line. Multiplying $2x - 3y$ by $7y$, we obtain the second product $-14xy + 21y^2$, which we write under the first product, the terms being so arranged that like terms will be in vertical columns. Adding these partial products, we obtain the required product.

Example 2. Multiply $x^2 - 2x + x^3 + 1$ by $2 - x + x^2$.

Arranging both multiplicand and multiplier according to the descending powers of x, we have

$$\begin{array}{llllll}
x^3 & + x^2 & - 2x & + 1 & & \\
x^2 & - x & + 2 & & & \\
\hline
x^5 & + x^4 & - 2x^3 & + x^2 & & \\
 & - x^4 & - x^3 & + 2x^2 & - x & \\
 & & 2x^3 & + 2x^2 & - 4x & + 2 \\
\hline
x^5 & & - x^3 & + 5x^2 & - 5x & + 2
\end{array}$$

Although not necessary, it will be found convenient to arrange both the multiplicand and the multiplier either according to the descending, or according to the ascending, powers of the same letter. The like terms of the partial products will thus be more readily arranged in vertical columns.

Exercise 14.

Multiply

1. $x + 2y$ by $x - 2y$.
2. $2x + 3y$ by $3x - 2y$.
3. $a - 3b$ by $a + 3b$.
4. $x + 7$ by $x - 6$.
5. $3x - 7$ by $2x - 1$.
6. $2x - 4$ by $2x + 6$.
7. $2y + 5b$ by $3y - 4b$.
8. $2m^2 + 5n^2$ by $2m^2 - n^2$.
9. $3m^2 - 1$ by $3m^2 + 1$.
10. $-x + 7$ by $x - 7$.
11. $-x - 16$ by $-x + 16$.
12. $-x + 21$ by $x - 21$.
13. $2a + \frac{1}{3}b$ by $3a + \frac{1}{2}b$.
14. $\frac{1}{4}a - \frac{1}{3}b$ by $\frac{1}{3}a - \frac{1}{4}b$.
15. $ax - by$ by $ax + by$.
16. $x^2 + x + 1$ by $x - 1$.
17. $a^2 + ab + b^2$ by $a - b$.
18. $a^2 - ab + b^2$ by $a + b$.

19. $x^4 - x^2y^2 + y^4$ by $x^2 + y^2$.

20. $a^2 - ab + b^2$ by $a^2 + ab + b^2$.

21. $a^2 - 2ax + 4x^2$ by $a^2 + 2ax + 4x^2$.

22. $10a^2 + 12ab + 9b^2$ by $4a - 3b$.

23. $a^2x - ax^2 + x^3 - a^3$ by $x + a$.

24. $x^2 + x - 2$ by $x^2 + x - 6$.

25. $2x^3 - 3x^2 + 2x$ by $2x^2 + 3x + 2$.

26. $a^3 + 2a^2b + 2ab^2$ by $a^2 - 2ab + 2b^2$.

27. $x^2 - 3xy - y^2$ by $-x^2 + xy + y^2$.

28. $x^2 - 2xy + y^2$ by $x^2 + 2xy + y^2$.

29. $27x^3 - 36ax^2 + 48a^2x - 64a^3$ by $3x + 4a$.

30. $ab + cd + ac + bd$ by $ab + cd - ac - bd$.

31. $x^{12} - x^9y^2 + x^6y^4 - x^3y^6 + y^8$ by $x^3 + y^2$.

32. $-2x^3y + y^4 + 3x^2y^2 + x^4 - 2xy^3$ by $x^2 + 2xy + y^2$.

33. $a^2 + b^2 + c^2 - bc - ca - ab$ by $a + b + c$.

34. $x^{n+2} + 2x^{n+1} - 3x^n - 1$ by $x + 1$.

35. $-ax^2 + 3axy^2 - 9ay^4$ by $-ax - 3ay^2$.

36. $-x^3y + y^4 + x^2y^2 + x^4 - xy^3$ by $x + y$.

37. $\frac{1}{2}a^2 + \frac{1}{3}a + \frac{1}{4}$ by $\frac{1}{2}a - \frac{1}{3}$.

38. $\frac{1}{2}x^2 - 2x + \frac{3}{2}$ by $\frac{1}{2}x + \frac{1}{3}$.

39. $\frac{2}{3}x^2 + xy + \frac{3}{2}y^2$ by $\frac{1}{3}x - \frac{1}{2}y$.

40. $\frac{1}{2}x^2 - \frac{2}{3}x - \frac{3}{4}$ by $\frac{1}{2}x^2 + \frac{2}{3}x - \frac{3}{4}$.

41. $\frac{3}{2}x^2 - ax - \frac{2}{3}a^2$ by $\frac{3}{4}x^2 - \frac{1}{2}ax + \frac{1}{3}a^2$.

42. $\frac{2}{3}ax + \frac{2}{3}x^2 + \frac{1}{3}a^2$ by $\frac{3}{4}a^2 + \frac{3}{2}x^2 - \frac{3}{2}ax$.

68. Removal of Signs of Grouping.

EXAMPLE. By removing the signs of grouping, simplify

$$42 - 5[-12x - 3\{-15x + 3(8 - \overline{7 - 3x})\}].$$

$$\begin{aligned} \text{The expression } &= 42 - 5[-12x - 3\{-15x + 3(3x + 1)\}] \\ &= 42 - 5[-12x - 3\{-6x + 3\}] \\ &= 42 - 5[6x - 9] \\ &= 87 - 30x. \end{aligned}$$

Exercise 15.

By removing the signs of grouping, simplify

1. $3b - \{5a - [6a + 2(10 - b)]\}$.
2. $a - (b - c) - [a - b - c - 2\{b + c\}]$.
3. $8(b + c) - [-\{a - b - 3(c - b + a)\}]$.
4. $2(3b - 5a) - 7[a - 6\{2 - (5a - b)\}]$.
5. $6\{a - 2[b - 3(c + d)]\} - 4\{a - 3[b - 4(c + d)]\}$.
6. $5\{a - 2[a - 2(a + x)]\} - 4\{a - 2[a - 2(a + x)]\}$.
7. $-10\{a - 6[a - (b - c)]\} + 60\{b - (c + a)\}$.
8. $-3\{-2[-4(-a)]\} + 5\{-2[-2(-a)]\}$.
9. $-2\{-1[-(x - y)]\} + \{-2[-(x - y)]\}$.

Multiply together the following expressions, and arrange each product according to the powers of x:

10. $ax^2 + bx + 1$ and $cx + 2$.
11. $ax^2 - 2bx + 3c$ and $x - 1$.
12. $x^3 + ax^2 - bx - c$ and $x^3 - ax^2 - bx + c$.
13. $ax^3 - x^2 + 3x - b$ and $ax^3 + x^2 + 3x + b$.
14. $x^4 - ax^3 - bx^2 + cx + d$ and $x^4 + ax^3 - bx^2 - cx + d$.

CHAPTER VI.

DIVISION.

69. Any indicated quotient is called a **Fraction.** A quotient is often indicated by placing the dividend over the divisor with a line between them.

Thus $a \div b$, and $\frac{a}{b}$, or a/b, are but different ways of indicating that a is to be divided by b.

Each of these expressions is a fraction, a being the *dividend*, and b the *divisor*. The dividend and divisor of a fraction are often called its *numerator* and *denominator* respectively.

When the dividend or divisor consists of more than one term, the dividing line in a fraction serves as a sign both of *division* and of *grouping*.

Thus in the fraction $\frac{a-b}{c+d}$ the dividing line takes the place of both the sign of division and the two parentheses in the form $(a-b)\div(c+d)$.

70. The **Reciprocal** of any number is 1 divided by that number.

Thus the reciprocal of d is $1 \div d$.

71. *Any number and its reciprocal have the same quality.*

For $(1 \div d) \times d = +1$; and if the product of any two factors is positive, both factors are positive, or both are negative.

72. *If the product of two numbers is* $+1$, *either number equals the reciprocal of the other.*

For if $ad = 1$, then $a = 1 \div d$, or $d = 1 \div a$.

73. *Dividing by a number is equivalent to multiplying by its reciprocal;* that is,

$$D \div d \equiv D \times (1 \div d) \text{ or } \frac{D}{d} \equiv D \times \frac{1}{d}. \qquad (1)$$

For either member of (1) multiplied by d, equals D. Hence, by § 48, (1) is an identity.

Thus $9 \div 3 \equiv 9 \times (1 \div 3)$ and $a \div b \equiv a \times (1 \div b)$.

Since $a \div b \equiv a \times (1 \div b) \equiv 1 \div b \times a$, the algebraic definition of a fraction includes all arithmetical fractions.

74. Laws of Division. By § 73, a *divisor* can be expressed as a *factor* of like quality; hence the laws of *order*, *grouping*, and *quality* in multiplication hold true for division also. Hence,

Like signs in dividend and divisor give + *in the quotient, and unlike signs give* −.

Thus the quality of the quotient of -14 by -7 is +; while that of -15 by 3, or that of 16 by -4 is −.

From the laws of *order* and *grouping*, it follows that a product is multiplied or divided by any number by multiplying or dividing any one of its factors by that number.

Thus $15 \div 7 \times 21 \div 5 = 15 \div 5 \times 21 \div 7 = 3 \times 3 = 9$.

A similar advantage is frequently gained by changing the order and grouping of factors and divisors.

EXAMPLE. Divide -18 by -3.

By the law of quality the quotient is +, and by arithmetic its arithmetical value is 6; hence,

$$(-18) \div (-3) = +6.$$

To verify the result, we have $(-3) \times (+6) = -18$.

75. Product of Fractions. *The product of two or more fractions is equal to the product of their dividends divided by the product of their divisors, and conversely;* that is,

$$\frac{a}{x}\cdot\frac{b}{y}\cdot\frac{c}{z}\equiv\frac{abc}{xyz}. \qquad (1)$$

For $$\frac{a}{x}\cdot\frac{b}{y}\cdot\frac{c}{z}(xyz)\equiv\frac{a}{x}x\cdot\frac{b}{y}y\cdot\frac{c}{z}z\equiv abc. \qquad (2)$$

Dividing the first and the last member of (2) by xyz, we obtain (1).

76. Law of Exponents. *The quotient of the* m*th power of any base divided by its* n*th power equals the* (m − n)*th power of that base.*

That is, $$a^m \div a^n = a^{m-n}.$$

For the quotient, a^{m-n}, into the divisor, a^n, equals a^{m-n+n}, or a^m, the dividend.

Thus $a^6 \div a^4 = a^{6-4} = a^2$; for $a^2 \times a^4 = a^6$.

77. Zero as Exponent. *Any base with zero as an exponent is equal to* $+1$.

For by the law of exponents in § 76, we have

$$a^n \div a^n = a^{n-n} = a^0.$$

But $$a^n \div a^n = +1.$$

Hence, by Ax. 5, $$a^0 = +1.$$

This meaning of a^0 is consistent with the definition of a positive integral exponent in § 55; for by that definition a^0 denotes that a is to be used as a factor *no* times, and hence the coefficient of a, or $+1$, left unmodified, is the value of the expression a^0; that is, $a^0 = +1$ $a^0 = +1$.

Thus $5\,b^0 = 5$, and $7\,y^0 = 7$.

Example 1. Divide $20\,a^4b^5$ by $-5\,ab^3$.

$$\frac{20\,a^4b^5}{-5\,ab^3} = \frac{20}{-5}\cdot\frac{a^4}{a}\cdot\frac{b^5}{b^3} \qquad \S\ 75$$

$$= -4\,a^3b^2. \qquad \S\S\ 74,\ 76$$

Example 2. Divide $-3a^7b^5c$ by $7a^5b^2$.

$$\frac{-3a^7b^5c}{7a^5b^2}=\frac{-3}{7}\cdot\frac{a^7}{a^5}\cdot\frac{b^5}{b^2}\cdot\frac{c}{1}$$
$$=-\tfrac{3}{7}a^2b^3c.$$

Example 3. Divide $-5a^2b^4x^3$ by $11a^2b^2x^3$.

$$\frac{-5a^2b^4x^3}{11a^2b^2x^3}=\frac{-5}{11}\cdot\frac{a^2}{a^2}\cdot\frac{b^4}{b^2}\cdot\frac{x^3}{x^3}$$
$$=-\tfrac{5}{11}b^2.$$

78. To divide one monomial by another. By § 75 we have the following rule:

Divide the numerical coefficient of the dividend by that of the divisor, observing the law of quality; then divide the literal factors, observing the law of exponents.

Thus $-84a^5x^3\div(12a^4x)=-7ax^2,$

and $77a^2x^3y^4\div(-7ax^2y)=-11axy^3.$

Exercise 16.

Divide

1. $+27$ by $+9$.
2. -27 by -9.
3. -32 by $+4$.
4. $+14$ by -2.
5. $-72a^2$ by $-9a$.
6. $84a^3$ by $-7a^2$.
7. $-35x^3$ by $7x^3$.
8. $4a^2b^3c^3$ by $-ab^2c^2$.
9. $-12a^6b^6c^6$ by $-3a^4bc^2$.
10. $84x^2y^6z^5$ by $-7xy^2z^3$.
11. $28a^4b^3$ by $-4a^3b$.
12. $-35a^3b^3$ by $5ab$.
13. $-16x^2y^6$ by $-4xy^4$.
14. $36m^{10}a^{12}$ by $9m^6a^9$.
15. $96a^4x^8z^4$ by $12a^3x^5z^2$.
16. $-256x^8y^9z^{11}$ by $-8x^4yz^6$
17. $84a^2b^7c^5$ by $14ab^2c^3$.
18. $16b^2yx^2$ by $-2xy$.
19. $-50y^3x^3$ by $-5x^3y$.
20. $144a^7z^9$ by $-24a^4z^8$.
21. $-3x^{m+2}$ by $5x^{m+1}$.
22. $-4x^{m+n}y^{m+n}$ by $7x^my^m$.

79. Distributive Law. *The quotient of one expression divided by another equals the sum of the results obtained by dividing the parts of the first expression by the second;* that is,

$$\frac{a+b+c+\cdots}{x}=\frac{a}{x}+\frac{b}{x}+\frac{c}{x}+\cdots. \qquad \text{(C}'\text{)}$$

For

$$\frac{a+b+c+\cdots}{x}=(a+b+c+\cdots)\frac{1}{x} \qquad \S\ 73$$

$$=a\frac{1}{x}+b\frac{1}{x}+c\frac{1}{x}+\cdots \qquad \S\ 54$$

$$=\frac{a}{x}+\frac{b}{x}+\frac{c}{x}+\cdots. \qquad \S\ 73$$

80. To divide a polynomial by a monomial.

By the distributive law we have the following rule:

Divide each term of the polynomial by the monomial, and add the several quotients.

EXAMPLE 1. Divide $12x^3-5ax^2-2a^2x$ by $3x$.

$$\frac{12x^3-5ax^2-2a^2x}{3x}=\frac{12x^3}{3x}+\frac{-5ax^2}{3x}+\frac{-2a^2x}{3x} \qquad \S\ 79$$

$$=4x^2-\tfrac{5}{3}ax-\tfrac{2}{3}a^2.$$

EXAMPLE 2. Divide $12a^3+9a^4-6a^5$ by $-3a^2$.

$$\frac{12a^3+9a^4-6a^5}{-3a^2}=\frac{12a^3}{-3a^2}+\frac{9a^4}{-3a^2}+\frac{-6a^5}{-3a^2}$$

$$=-4a-3a^2+2a^3.$$

Exercise 17.

Divide

1. $5x^2-7ax+4x$ by x.
2. $x^6-7x^5+4x^4$ by x^2.
3. $10x^7-8x^6+3x^4$ by x^3.
4. $27x^6-36x^5$ by $9x^5$.
5. $15x^5-25x^4$ by $-5x^3$.
6. $-24x^6-32x^4$ by $-8x^3$.
7. $a^3-a^2b-a^2b^2$ by a^2.
8. $a^2-ab-ac$ by $-a$.
9. $xy-x^2-ax$ by $-x$.
10. $3x^3-9x^2y^2$ by $-3x$.

11. $4a^4b^4 - 8a^3b^3 + 6ab^3$ by $-2ab$.
12. $-3x^2 + \frac{9}{2}xy - 6xz$ by $-\frac{3}{2}x$.
13. $-\frac{5}{2}x^2 + \frac{5}{3}xy + \frac{10}{3}x$ by $-\frac{5}{6}x$.
14. $\frac{1}{2}x^5y^2 - 3x^3y^4 - 5x^4y^3$ by $-\frac{3}{2}x^3y^2$.
15. $\frac{1}{4}a^2x - \frac{1}{16}abx - \frac{3}{8}acx$ by $\frac{3}{8}ax$.
16. $-2a^5x^3 + \frac{7}{2}a^4x^4$ by $\frac{7}{3}a^3x$.

81. To divide one polynomial by another.

Let it be required to divide

$$2x^2y^2 - x^3y + x^4 - xy^3 + y^4 \text{ by } x^2 - xy + y^2.$$

First arrange both the dividend and divisor according to the descending powers of x, for convenience placing the divisor to the right of the dividend as below:

$$\begin{array}{l|l}
x^4 - x^3y + 2x^2y^2 - xy^3 + y^4 & x^2 - xy + y^2 \text{ Divisor.} \\
\underline{x^4 - x^3y + x^2y^2} & \underline{x^2 + y^2} \text{ Quotient.} \\
\qquad\qquad\quad x^2y^2 - xy^3 + y^4 & \\
\qquad\qquad\quad \underline{x^2y^2 - xy^3 + y^4} &
\end{array}$$

Now x^4, the term of the highest degree in x in the dividend, must be the product of the terms of highest degree in the divisor and the quotient; hence, the first term of the quotient must be $x^4 \div x^2$, or x^2. Multiply the divisor by x^2 and subtract the result from the dividend. The remainder, $x^2y^2 - xy^3 + y^4$, must be the product of the divisor by the other terms of the quotient; hence, x^2y^2, the first term of the remainder, must be the product of the first term of the divisor and the second term of the quotient. Therefore the second term of the quotient is $x^2y^2 \div x^2$, or y^2. Multiplying the divisor by y^2 and subtracting the result from $x^2y^2 - xy^3 + y^4$, we have no remainder. Hence $x^2 + y^2$ is the required quotient.

By the process the dividend has been separated into the two parts $x^4 - x^3y + x^2y^2$ and $x^2y^2 - xy^3 + y^4$; that is,

$$x^4 - x^3y + 2x^2y^2 - xy^3 + y^4 = (x^4 - x^3y + x^2y^2) + (x^2y^2 - xy^3 + y^4).$$

$$\therefore \frac{x^4 - x^3y + 2x^2y^2 - xy^3 + y^4}{x^2 - xy + y^2} = \frac{x^4 - x^3y + x^2y^2}{x^2 - xy + y^2} + \frac{x^2y^2 - xy^3 + y^4}{x^2 - xy + y^2}$$
$$= x^2 + y^2.$$

If we had arranged both the dividend and divisor according to the ascending powers of x, the quotient would have been obtained in the form $y^2 + x^2$.

Hence, to divide one polynomial by another, we have the following rule:

Arrange both the dividend and divisor according to the ascending or descending powers of some common letter.

Divide the first term of the dividend by the first term of the divisor: the result will be the first term of the quotient.

Multiply the divisor by the first term of the quotient; and subtract the product from the dividend. Treat the remainder, if any, as a new dividend, and so on.

Example 1. Divide $3a^3 - 4a^2 + 2a - 1$ by $1 - a$.

$$\begin{array}{r|l}
3a^3-4a^2+2a-1 & -a+1 \\ \cline{2-2}
3a^3-3a^2\phantom{{}+2a-1} & -3a^2+a-1 \\ \hline
-a^2+2a-1 & \\
-a^2+a\phantom{{}-1} & \\ \cline{1-1}
a-1 & \\
a-1 & \\ \cline{1-1}
\end{array}$$

Here, $\dfrac{3a^3-4a^2+2a-1}{-a+1} = \dfrac{3a^3-3a^2}{-a+1} + \dfrac{-a^2+a}{-a+1} + \dfrac{a-1}{-a+1}$

$$= -3a^2 + a - 1.$$

Example 2. Divide $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

$$\begin{array}{r|l}
x^4 \phantom{{}-x^3y} + x^2y^2 \phantom{{}-xy^3} + y^4 & x^2-xy+y^2 \\ \cline{2-2}
x^4 - x^3y + x^2y^2 \phantom{{}-xy^3+y^4} & x^2+xy+y^2 \\ \hline
x^3y \phantom{{}-x^2y^2-xy^3} + y^4 & \\
x^3y - x^2y^2 + xy^3 \phantom{{}+y^4} & \\ \cline{1-1}
x^2y^2 - xy^3 + y^4 & \\
x^2y^2 - xy^3 + y^4 & \\ \cline{1-1}
\end{array}$$

Example 3. Divide $16a^4-1$ by $2a-1$.

$$\begin{array}{rl}
16a^4 \qquad\qquad\qquad\qquad -1 & \underline{\big|\,2a-1\qquad\qquad\qquad} \\
\underline{16a^4-8a^3 \qquad\qquad\qquad\quad\;\;} & \big|\,8a^3+4a^2+2a+1 \\
8a^3 \qquad\qquad -1 & \\
\underline{8a^3-4a^2 \qquad\qquad} & \\
4a^2 \qquad -1 & \\
\underline{4a^2-2a \qquad} & \\
2a-1 & \\
\underline{2a-1} &
\end{array}$$

Exercise 18.

Divide

1. x^2+3x+2 by $x+1$.
2. $a^2-11a+30$ by $a-5$.
3. $x^2-7x+12$ by $x-3$.
4. $3x^2+10x+3$ by $x+3$.
5. $5x^2+11x+2$ by $x+2$.
6. $5x^2+16x+3$ by $x+3$.
7. $2x^2+11x+5$ by $2x+1$.
8. $2x^2+17x+21$ by $2x+3$.
9. $4x^2+23x+15$ by $4x+3$.
10. $6x^2-7x-3$ by $2x-3$.
11. $12a^2-7ax-12x^2$ by $3a-4x$.
12. $15a^2+17ax-4x^2$ by $3a+4x$.
13. $12a^2-11ac-36c^2$ by $4a-9c$.
14. $60x^2-4xy-45y^2$ by $10x-9y$.
15. $-4xy-15y^2+96x^2$ by $12x-5y$.
16. $100x^3-3x-13x^2$ by $3+25x$.
17. $16-96x+216x^2-216x^3+81x^4$ by $2-3x$.
18. $x^3-x^2-9x-12$ by x^2+3x+3.
19. $2y^3-3y^2-6y-1$ by $2y^2-5y-1$.
20. $6m^3-m^2-14m+3$ by $3m^2+4m-1$.
21. $6a^5-13a^4+4a^3+3a^2$ by $3a^3-2a^2-a$.

22. $x^4 + x^3 + 7x^2 - 6x + 8$ by $x^2 + 2x + 8$.

23. $a^4 - a^3 - 6a^2 + 15a - 9$ by $a^2 + 2a - 3$.

24. $a^4 + 6a^3 + 13a^2 + 12a + 4$ by $a^2 + 3a + 2$.

25. $2x^4 - x^3 + 4x^2 + 4x - 3$ by $x^2 - x + 3$.

26. $x^5 - 5x^4 + 9x^3 - 6x^2 - x + 2$ by $x^2 - 3x + 2$.

27. $x^5 - 4x^4 + 3x^3 + 3x^2 - 3x + 2$ by $x^2 - x - 2$.

28. $30x^4 + 11x^3 - 82x^2 - 12x + 48$ by $2x - 4 + 3x^2$.

29. $69y - 18 - 71y^3 + 28y^4 - 35y^2$ by $4y^2 - 13y + 6$.

30. $6k^5 - 15k^4 + 4k^3 + 7k^2 - 7k + 2$ by $3k^3 - k + 1$.

31. $2x^3 - 8x + x^4 + 12 - 7x^2$ by $x^2 + 2 - 3x$.

32. $x^5 - 2x^4 - 7x^3 + 19x^2 - 10x$ by $x^3 - 7x + 5$.

33. $14x^4 + 45x^3y + 78x^2y^2 + 45xy^3 + 14y^4$ by $2x^2 + 5xy + 7y^2$.

34. $x^5 + x^4y - x^3y^2 + x^3 - 2xy^2 + y^3$ by $x^2 + xy - y^2$.

35. $x^7 - 2y^{14} - 7x^5y^4 - 7xy^{12} + 14x^3y^8$ by $x - 2y^2$.

36. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

$$
\begin{array}{l|l}
a^3 - 3abc + b^3 + c^3 & a + b + c \\
\cline{2-2}
a^3 + a^2b + a^2c & a^2 - ab - ac + b^2 - bc + c^2 \\
\hline
\quad -a^2b - a^2c - 3abc + b^3 + c^3 & \\
\quad \underline{-a^2b - ab^2 - abc} & \\
\qquad\qquad -a^2c + ab^2 - 2abc + b^3 + c^3 & \\
\qquad\qquad \underline{-a^2c \quad - abc - ac^2} & \\
\qquad\qquad\quad ab^2 - abc + ac^2 + b^3 + c^3 & \\
\qquad\qquad\quad \underline{ab^2 \qquad\qquad + b^3 + b^2c} & \\
\qquad\qquad\qquad -abc + ac^2 - b^2c + c^3 & \\
\qquad\qquad\qquad \underline{-abc \qquad - b^2c - bc^2} & \\
\qquad\qquad\qquad\qquad ac^2 + bc^2 + c^3 & \\
\qquad\qquad\qquad\qquad \underline{ac^2 + bc^2 + c^3} &
\end{array}
$$

Here we arranged the dividend and divisor according to the powers of a, and gave b precedence to c throughout.

37. $x^3 + y^3 - z^3 + 3xyz$ by $x + y - z$.

38. $8x^3 - y^3 + z^3 + 6xyz$ by $y - z - 2x$.

39. $27a^3 - 8b^3 + c^3 + 18abc$ by $3a - 2b + c$.

40. $a^3 + 3a^2b + 3ab^2 + b^3 + c^3$ by $a + b + c$.

41. $a^4 + b^4 + c^4 - 2b^2c^2 - 2a^2c^2 - 2a^2b^2$ by $a + b + c$.

42. $\frac{1}{4}x^3 + \frac{1}{72}xy^2 + \frac{1}{12}y^3$ by $\frac{1}{2}x + \frac{1}{3}y$.

$$\begin{array}{l|l}
\frac{1}{4}x^3 + \frac{1}{72}xy^2 + \frac{1}{12}y^3 & \frac{1}{2}x + \frac{1}{3}y \\ \cline{2-2}
\underline{\frac{1}{4}x^3 + \frac{1}{6}x^2y} & \frac{1}{2}x^2 - \frac{1}{3}xy + \frac{1}{4}y^2 \\ \hline
\quad - \frac{1}{6}x^2y + \frac{1}{72}xy^2 + \frac{1}{12}y^3 & \\
\quad \underline{- \frac{1}{6}x^2y - \frac{1}{9}xy^2} & \\
\qquad\qquad \frac{1}{8}xy^2 + \frac{1}{12}y^3 & \\
\qquad\qquad \underline{\frac{1}{8}xy^2 + \frac{1}{12}y^3} &
\end{array}$$

43. $\frac{1}{8}a^3 - \frac{9}{4}a^2x + \frac{27}{2}ax^2 - 27x^3$ by $\frac{1}{2}a - 3x$.

44. $\frac{1}{27}a^3 - \frac{1}{12}a^2 + \frac{1}{16}a - \frac{1}{64}$ by $\frac{1}{3}a - \frac{1}{4}$.

45. $\frac{3}{4}a^2c^3 + \frac{6}{125}a^5$ by $\frac{1}{5}a^2 + \frac{1}{2}ac$.

46. $\frac{9}{16}a^4 - \frac{3}{4}a^3 - \frac{7}{4}a^2 + \frac{4}{3}a + \frac{16}{9}$ by $\frac{3}{2}a^2 - \frac{8}{3} - a$.

47. $36x^2 + \frac{1}{9}y^2 + \frac{1}{4} - 4xy - 6x + \frac{1}{3}y$ by $6x - \frac{1}{3}y - \frac{1}{2}$.

48. $\frac{8}{27}a^5 - \frac{243}{512}ax^4$ by $\frac{2}{3}a - \frac{3}{4}x$.

82. In each example given above the divisor has been exactly contained in the dividend. When the division is not exact, the work should be carried on until the remainder is of lower degree than the divisor. The *terms* of the quotient thus obtained are sometimes called the quotient. The true quotient, however, is the sum of these terms plus the quotient of the remainder by the divisor.

For example, $$\frac{x^2 + 3ax + a^2}{x + a} = x + 2a - \frac{a^2}{x + a}. \tag{1}$$

The true quotient is the entire second member of (1), of which the last term is a fraction. Sometimes, however, it is convenient to call $x + 2a$ the quotient and $-a^2$ the remainder.

83. *Multiplying or dividing both the dividend and the divisor by the same number does not change the value of the fraction or quotient.*

For, since $m \div m = 1$, by § 75 we have

$$\frac{a}{x} = \frac{a}{x} \cdot \frac{m}{m} = \frac{am}{xm}, \text{ or } \frac{am}{xm} = \frac{a}{x}.$$

Example. Simplify the fraction $\frac{15\,ax^3y^2}{25\,a^2xy^6}$.

The factor $5\,axy^2$ is common to both dividend and divisor; hence

$$\frac{15\,ax^3y^2}{25\,a^2xy^6} = \frac{(15\,ax^3y^2) \div (5\,axy^2)}{(25\,a^2xy^6) \div (5\,axy^2)}$$
$$= \frac{3\,x^2}{5\,ay^4}.$$

84. *The reciprocal of a fraction is equal to the fraction inverted; that is, the reciprocal of $a \div b$ equals $b \div a$.*

For $$\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1;$$

hence, by § 72, $a \div b$ and $b \div a$ are reciprocals of each other.

Thus the reciprocal of $\frac{7}{8}$ is $\frac{8}{7}$.

85. Dividing by a Fraction. By § 73, dividing by a fraction is equivalent to multiplying by its reciprocal; hence

To divide by a fraction,

Multiply by the fraction inverted.

Example. Divide $\frac{3\,a}{5\,b}$ by $\frac{2\,x}{7\,y}$.

$$\frac{3\,a}{5\,b} \div \frac{2\,x}{7\,y} = \frac{3\,a}{5\,b} \times \frac{7\,y}{2\,x} = \frac{21\,ay}{10\,bx}.$$

86. To multiply a fraction by any number,

Multiply the dividend, or divide the divisor, by that number.

For $$\frac{a}{b}\cdot\frac{m}{1}=\frac{am}{b}=\frac{a}{b\div m}. \qquad \S\ 83$$

87. To divide a fraction by any number,

Divide the dividend, or multiply the divisor, by that number.

For $$\frac{a}{b}\div m=\frac{a}{b}\times\frac{1}{m} \qquad \S\ 73$$

$$=\frac{a}{bm}=\frac{a\div m}{b}. \qquad \S\ 83$$

From the preceding principles it follows that in Algebra fractions are simplified, added, subtracted, multiplied, and divided in the same way as in arithmetic.

88. From the law of quality it follows that

Changing the quality of both the dividend and divisor does not change the value of the fraction or quotient.

Changing the quality of either dividend or divisor changes the quality of the fraction or quotient.

89. The sign before a fraction denotes the quality of its numerical coefficient; hence

The sign before a fraction may be changed if the quality of its dividend or divisor be changed.

For changing the quality of both the fraction and its coefficient will not change the quality of the expression.

Thus $$\frac{a}{b}=+1\frac{a}{b}=-1\frac{-a}{b}, \text{ or } -\frac{-a}{b}.$$

Again, $$-\frac{abc}{xyz}=\frac{(-a)(-b)(-c)}{xyz}, \text{ or } \frac{abc}{-xyz}.$$

Example. Simplify $-\dfrac{-5ab^3}{15a^3b^2}$.

$$-\frac{-5ab^3}{15a^3b^2}=\frac{5ab^3}{15a^3b^2}=\frac{b}{3a^2}.$$

Exercise 19.

Simplify each of the following expressions:

1. $-\dfrac{-25}{125}$.

2. $-\dfrac{125}{-625}$.

3. $\dfrac{-32}{-48}$.

4. $\dfrac{-a^2b}{-ab^2}$.

5. $-\dfrac{-4a^2}{16ab}$.

6. $-\dfrac{2xy^2}{-5x^2y}$.

7. $\dfrac{-2a^2bc^2}{-4a^4b^3c}$.

8. $\dfrac{-8a^4b^2c^5}{16a^2b^3c^4}$.

9. $-\dfrac{-15x^3y^2z^7}{25x^4yz^5}$.

10. $-\dfrac{14ax^3y^2}{-21a^2xy^6}$.

11. $\dfrac{2a}{3c}\times\dfrac{3c}{4a}$.

12. $\dfrac{2a}{3b}\times\dfrac{6c}{5a}\times\dfrac{5x}{2y}$.

13. $\dfrac{2a}{5b}\times\dfrac{5c}{x}\times\dfrac{x^2b}{y}$.

14. $\dfrac{2a^2}{bc}\div\dfrac{3abx}{c^2y}$.

15. $\dfrac{3axy}{5b^2}\div\dfrac{6a^2y^2}{10bx^2}$.

16. $\dfrac{b}{a}\times\dfrac{x}{c}\div\dfrac{c}{a}$.

17. $\dfrac{a^2}{b^2}\times\dfrac{x^2}{y^2}\div\dfrac{a^2}{y^2}$.

18. $\dfrac{3a^2}{4b}\times\dfrac{2c}{7ax}\div\dfrac{3a}{7bx^2}$.

19. $\dfrac{5y^2}{7a^3}\times\dfrac{21c^3}{4ax}\div\dfrac{35c^2y}{7a^3x}$.

20. $\dfrac{2b}{3a}\div\dfrac{x}{y}\times\dfrac{9a^2}{4b^2}$.

21. Reduce the fractions $\dfrac{2x}{a^2}$, $\dfrac{b}{3x^2}$, and $\dfrac{c}{2ax}$ to equivalent fractions having a common denominator.

The simplest expression that will contain each denominator is evidently $6a^2x^2$. Multiplying both the numerator and denominator of each fraction by the factor which is required to make its denominator $6a^2x^2$, we have the equivalent fractions

$$\frac{12x^3}{6a^2x^2},\ \frac{2a^2b}{6a^2x^2},\ \frac{3acx}{6a^2x^2}.$$

The process is precisely the same as in arithmetic.

Reduce to common denominators the fractions:

22. $\frac{2x}{a}, \frac{y}{2a}.$
24. $\frac{4a}{5b}, \frac{3a}{10c}.$
26. $\frac{a}{bc}, \frac{b}{ca}.$
23. $\frac{n}{2x}, \frac{n}{3x}.$
25. $\frac{a}{b}, \frac{c}{d}, \frac{x}{y}.$
27. $\frac{3a}{7b}, \frac{5b}{21c}.$

90. Fractional and Integral Terms. If a term contains a *letter* as a *divisor*, the term is said to be *Fractional;* otherwise it is said to be *Integral.*

Thus x^2, ab, $\frac{1}{2}by^2$, and $-\frac{3}{4}az^2$ are *integral* terms; while $a \div b$, $(a-b) \div c$, and $x \div (a-y)$ are *fractional* terms. An integral term may have either an *integral* or *fractional* value; so also may a fractional term. Thus, for $a = 8$ and $b = 2$, $ab = 16$ and $a \div b = 4$; for $a = 5$ and $b = 3$, $ab = 15$ and $a \div b = \frac{5}{3}$; for $a = \frac{1}{2}$ and $b = \frac{1}{6}$, $ab = \frac{1}{12}$ and $a \div b = 3$.

This classification of terms has no reference to their numerical value, but simply to the presence or absence of a *literal divisor.*

91. An **Integral Expression** is one of which the terms are all integral; as, $x^2 + 3ax + 5c^2y$ and $\frac{1}{8}ax - \frac{1}{7}cy$.

A **Fractional Expression** is one of which the terms are all fractional; as, $\frac{a}{b} + \frac{cx^2}{y} + \frac{dz}{m}$ and $\frac{a^2}{2x} - \frac{y^2}{5c^3}$.

An expression may be integral with respect to some of its letters, and fractional with respect to others.

Thus $\frac{ax^3}{c} - \frac{x^2}{b} + \frac{1}{2}x$ is integral with respect to x, but fractional with respect to b or c.

92. A **Mixed Expression** is one that contains both integral and fractional terms; as, $\frac{1}{2}x^3 - c \div (a-b)$.

93. A **Rational Expression** is one which involves only additions, subtractions, multiplications, divisions, or powers of its number symbols.

All the expressions heretofore considered are rational expressions.

94. **Degree.** An integral rational term that contains n literal factors is said to be of n *Dimensions*, or of the n*th Degree*. But we often speak of the degree of a term with regard to some one of its letters.

Thus $5ab^2x^3$, which is of the sixth degree, is of the first degree *in* a, of the second degree *in* b, and of the third degree *in* x; also, $4abx^2y^3$, which is of the seventh degree, is of the second degree *in* x, of the third degree *in* y, and of the fifth degree *in* x *and* y. Again, the term $ax^3 \div c$, which is integral with respect to x, is of the third degree *in* x.

95. The **Degree of an Expression** is the degree of that term which is of the highest degree.

Thus $ax^2 + bx + bx + c$ is of the second degree in x; also, $ax^2y + bxy$ is of the second degree in x but of the third degree in x and y.

A *Linear* expression is one of the first degree.

A *Quadratic* expression is one of the second degree.

A *Cubic* expression is one of the third degree.

Thus $ax + b$ is a linear expression in x, and $ax + by + ca$ a linear expression in x and y. $ax^2 + bx + c$ is a quadratic expression in x, and $cxy + dx + ey + f$ is a quadratic expression in x and y. $ax^3 + bx^2 + c$ is a cubic expression in x, and $ax^2y + bxy^2$ is a cubic expression in x and y.

96. An expression is said to be **Homogeneous** when all its terms are of the same degree.

Thus $4x^3 + 5xy^2 - 7y^3$ is a homogeneous expression, each term being of the third degree.

CHAPTER VII.

LINEAR EQUATIONS.—ONE UNKNOWN NUMBER.

97. Degree of an Equation. When an equation of one unknown number, as x, contains only rational integral expressions in x, its *Degree* is that of the term of the highest degree in x.

Thus the equations $3x = 9$ and $ax + b = 0$ are of the first degree; $5x^2 - 7 = 0$ and $ax^2 + bx = c$ are of the second degree; $7x^3 - 4x = 7$ and $ax^3 + bx + cx = d$ are of the third degree; and so on.

A *Linear* equation is one of the first degree.

A *Quadratic* equation is one of the second degree.

A *Cubic* equation is one of the third degree.

EQUIVALENT EQUATIONS.

98. Two equations are said to be **Equivalent** when they have the same solutions (§ 31); that is, when the solutions of either include all the solutions of the other.

Thus $4x - 8 = 2 - x$ and $5x = 10$ are equivalent equations; for the only solution of either is 2.

In solving equations, frequent use is made of the following principles of equivalent equations:

99. *If for any expression in an equation an identical expression be substituted, the resulting equation will be equivalent to the first.*

That is, if A and B denote any two expressions involving x that are not identical, and $A \equiv C$, then

$$A = B \text{ and } C = B$$

will be equivalent equations.

For, since A and C are identical expressions, any value of x that will make either one equal to B will make the other equal to B.

Thus if in the equation

$$3(x-1) - \{3x - (2-x)\} = 5, \tag{1}$$

we remove the signs of grouping and combine like terms, we obtain the equivalent equation

$$-1 - x = 5. \tag{2}$$

Equations (1) and (2) are equivalent; for their first members, being identical expressions, will equal 5 for the same value of x.

100. *If the same expression be added to both members of an equation, the resulting equation will be equivalent to the first.*

That is, if M denote any expression, then

$$A = B \tag{1}$$

and

$$A + M = B + M \tag{2}$$

will be equivalent equations.

For any value of x that satisfies equation (1) must, by Axiom 1 of § 28, satisfy (2); and conversely, any value of x that satisfies (2) must, by Axiom 2, satisfy (1). Hence equations (1) and (2) are equivalent.

Thus if to each member of the equation

$$ax - b = cx - d \tag{1}$$

we add $-cx$ and $+b$, we will obtain the equivalent equation

$$ax - cx = b - d.$$

Adding $-cx$ to both members of equation (1) took the term cx from the second member and put it in the first member with its quality changed. Likewise, adding b to both members of (1) took the term $-b$ from the first member and put it in the second member with its quality changed; hence

If any term with its quality changed be transposed from one member of an equation to the other, the resulting equation will be equivalent to the first.

Thus $$3x^2 - 2a = 4x - 2x^3$$
and $$2x^3 + 3x^2 - 4x - 2a = 0$$
are equivalent equations.

101. An expression is said to be **unknown** or **known** according as it *contains* or does *not* contain the unknown letter or letters.

102. *If both members of an equation be multiplied by the same known expression, the resulting equation will be equivalent to the first.*

That is, if C denote any *known* expression, then

$$A = B \text{ and } CA = CB$$

will be equivalent equations.

For evidently any value of x that will render the expressions A and B equal will satisfy each equation, and no other value of x will satisfy either equation. Hence the two equations are equivalent.

Thus $$6x - 1 = 4x + 7$$
and $$5(6x - 1) = 5(4x + 17)$$
are equivalent equations. The only solution of either is 9.

If we multiply both members of an integral equation by an *unknown* expression, we introduce new solutions. Thus the equations

$$6x - 1 = 4x + 17 \qquad (1)$$
and $$(x - 4)(6x - 1) = (x - 4)(4x + 17) \qquad (2)$$

are not equivalent; for the only solution of (1) is 9, while (2) has the two solutions, 9 and 4. The solution 4 was introduced by mutliplying (1) by the unknown expression $x - 4$.

EXAMPLE. Solve $(5x - 12) \div 6 = (x - 3) \div 3.$ (1)

Multiply by 6, $5x - 12 = 2x - 6.$ (2)
Transpose, $5x - 2x = 12 - 6.$ (3)
Combine, $3x = 6.$ (4)
Divide by 3, $x = 2.$ (5)

Equation (2) is equivalent to (1) by § 102; (3) to (2) by § 100; (4) to (3) by § 99; (5) to (4) by § 102. Hence the solution of each of these equations is 2.

The student should keep in mind that

(i.) *Dividing* by any expression is equivalent to *multiplying* by its reciprocal.

(ii.) Multiplying both members by -1 will change the sign before each term of an equation.

(iii.) Multiplying both members by the denominator of any fractional term will clear the equation of that fraction.

EXAMPLE 1. Solve $(2 - 5x) \div a = (cx + 7) \div b.$ (1)

Multiplying both members of (1) by ab, we have

$$(2 - 5x)b = (cx + 7)a. \quad (2)$$

Remove ( ), $2b - 5bx = acx + 7a.$ (3)
Transpose, $-5bx - acx = 7a - 2b.$ (4)
Add like terms, $-(5b + ac)x = 7a - 2b.$ (5)
Divide by $-(5b + ac)$, $x = \frac{2b - 7a}{5b + ac}.$ (6)

Equation (2) is equivalent to (1) by § 102; (3) to (2) by § 99; (4) to (3) by § 100; (5) to (4) by § 99; and (6) to (5) by § 102. Hence the value of x in (6) is the solution of equation (1).

EXAMPLE 2. Solve $\frac{x+1}{4} - \frac{x-1}{3} = 1.$ (1)

Multiply by 12, $3(x + 1) - 4(x - 1) = 12.$ (2)
Remove ( ), $3x + 3 - 4x + 4 = 12.$ (3)
Transpose terms, $3x - 4x = 12 - 4 - 3.$ (4)
Add like terms, $-x = 5.$ (5)
Multiply by -1, $x = -5.$ (6)

Equation (2) is equivalent to (1) by § 102; (3) to (2) by § 99; (4) to (3) by § 100; (5) to (4) by § 99; and (6) to (5) by § 102. Hence -5 is the only solution of equation (1).

Example 3. Solve $(x-1)(x-2)+5=(x+1)^2$. (1)

Removing the signs of grouping, we have

$$x^2-3x+2+5=x^2+2x+1. \tag{2}$$

Transpose terms, $$x^2-x^2-3x-2x=1-2-5. \tag{3}$$

Add like terms, $$-5x=-6. \tag{4}$$

Divide by -5, $$x=\tfrac{6}{5}. \tag{5}$$

Equation (2) is equivalent to (1) by § 99; (3) to (2) by § 100; (4) to (3) by § 99; (5) to (4) by § 102. Hence $\frac{6}{5}$ is the solution of equation (1).

103. From the examples given above, it will be seen that the different steps in the process of solving a linear equation are the following:

First clear the equation of fractions and perform all indicated operations.

Then transpose the known terms to one member and the unknown terms to the other.

Next combine like terms, and divide both members by the coefficient of the unknown letter.

Note. In order to form the habit of clear and accurate thinking, the way in which each equation is derived from the preceding should always be indicated, and the reason for the equivalency of each two consecutive equations should be given.

Exercise 20.

Solve each of the following equations:

1. $5x-12=6x-8$. *Ans.* -4.
2. $7x+19=5x+7$. *Ans.* -6.
3. $x-(4-2x)=7(x-1)$.

4. $5(4-3x)=7(3-4x)$. *Ans.* $\frac{1}{13}$.

5. $2(x-3)=5(x+1)+2x-1$.

6. $4\,(1-x)+3(2+x)=13$.

7. $2\,(x-2)+3(x-3)+4(x-4)-20=0$. *Ans.* $5\frac{4}{9}$.

8. $2\,(x-1)-3(x-2)+4(x-3)+2=0$.

9. $5x+6(x+1)-7(x+2)-8(x+3)=0$.

10. $\frac{x}{5}-\frac{x}{4}=1$.

11. $\frac{x-1}{2}+\frac{x-2}{3}=3$.

12. $\frac{1}{2}(2-x)-\frac{1}{5}(5x+21)=x+3$. *Ans.* $-2\frac{12}{25}$.

13. $\frac{1}{2}(x+1)+\frac{1}{3}(x+2)+\frac{1}{4}(x+4)+8=0$.

14. $\frac{1}{2}(x-5)-\frac{1}{3}(x-4)=\frac{1}{2}(x-3)-(x-2)$.

15. $2x-[3-\{4x+(x-1)\}-5]=8$.

16. $(x-1)(x-2)=(x+3)(x-4)$.

17. $3x^2=(x+1)^2+(x+2)^2+(x+3)^2$.

18. $(x-2)(x-5)+(x-3)(x-4)=2(x-4)(x-5)$.

19. $5(x+1)^2+7(x+3)^2=12(x+2)^2$.

20. $(x-1)(x-4)=2x+(x-2)(x-3)$.

21. $\frac{1}{2}(x+\frac{1}{2})-\frac{1}{5}(2x-\frac{1}{2})+1\frac{1}{4}=0$.

22. $\frac{3x+5}{8}-\frac{21+x}{2}=5x-15$.

Multiply by 8, $3x+5-84-4x=40x-120$.

23. $0.5x+3.75=5.25x-1$.

Multiply by 100, $50x+375=525x-100$.

24. $0.25x+4-0.375x=0.2x-9$.

25. $0.15x+1.2-0.875x+0.375=0.0625x$.

26. $\frac{1}{2}(x+a+b)+\frac{1}{3}(x+a-b)=b$.

27. $\frac{1}{2}\,(a+x)+\frac{1}{3}(2a+x)+\frac{1}{4}(3a+x)=3a$.

28. $xa \div b + xb \div a = a^2 + b^2.$

29. $(a + bx)(b + ax) = ab(x^2 - 1).$ *Ans.* $-\dfrac{2ab}{a^2 + b^2}.$

30. $(a^2 + x)(b^2 + x) = (ab + x)^2.$

31. $(x + a + b)^2 + (x + a - b)^2 = 2x^2.$ *Ans.* $-\dfrac{a^2 + b^2}{2a}.$

32. $(x - a)(x - b) + (a + b)^2 = (x + a)(x + b).$

104. *Any linear equation has one, and only one, solution.*

By the preceding principles, any linear equation in x can be reduced to an equivalent equation of the form

$$ax = c. \tag{1}$$

Divide by a,
$$x = c \div a. \tag{2}$$

Since equations (1) and (2) are equivalent, $c \div a$ is the one and only solution of equation (1).

If $c = 0$, the solution becomes zero.

If c is not zero, then the smaller a is, the larger is the solution, $c \div a$.

105. Problems leading to Linear Equations. The student should again carefully read the suggestions in § 33, and then study the methods used in solving the following examples:

Example 1. A has \$80, and B has \$15. How much must A give to B in order that he may have just 4 times as much as B?

Let x = the *number* of dollars that A must give to B;
then $80 - x$ = the *number* of dollars that A will have left
and $15 + x$ = the *number* of dollars that B will have.

But A will then have 4 times as much as B; hence

$$80 - x = 4(15 + x). \tag{1}$$

From (1) $x = 4.$

Hence A must give \$4 to B.

Example 2. A man has 16 coins, some of which are half-dollars, and the rest dimes, and the coins altogether are worth $6. How many has he of each kind?

Let $x =$ the *number* of half-dollars;

then $16 - x =$ the *number* of dimes.

The x half-dollars are worth $\frac{1}{2}x$ dollars,

and the $16 - x$ dimes are worth $\frac{1}{10}(16 - x)$ dollars.

Now the coins altogether are worth $6; hence

$$\tfrac{1}{2}x + \tfrac{1}{10}(16 - x) = 6. \qquad (1)$$

From (1) $x = 11$, the *number* of half-dollars.

$\therefore\ 16 - x = 5$, the *number* of dimes.

Example 3. A father is 7 times as old as his son, and in 5 years he will be 4 times as old as his son. How old is each?

Let x years $=$ the son's age;

then $7x$ years $=$ the father's age.

Hence $x + 5$ years $=$ the son's age after 5 years,

and $7x + 5$ years $=$ the father's age after 5 years.

$$\therefore\ 7x + 5 = 4(x + 5). \qquad (1)$$

From (1) $x = 5. \quad \therefore\ 7x = 35.$

Hence the son is 5 years old, and the father 35.

Example 4. Divide 60 into two parts, so that 3 times the greater may exceed 100 by as much as 8 times the less falls short of 200.

Let $x =$ the greater part; then $60 - x =$ the less.

Three times the greater part is $3x$, and

$3x - 100 =$ the excess of $3x$ over 100.

Eight times the less part is $8(60 - x)$, and

$200 - 8(60 - x) =$ the defect of $8(60 - x)$ from 200.

But this excess and defect are equal; hence

$$3x - 100 = 200 - 8(60 - x). \qquad (1)$$

From (1) $x = 36$, the greater number.

$\therefore\ 60 - x = 24$, the less number.

Example 5. A could do a piece of work in 14 hours which B could do in 6 hours. A began the work, but after a time B took his place, and the whole work was finished in 10 hours from the beginning. How long did A work?

Let $x =$ the *number* of hours that A worked;
then $10 - x =$ the *number* of hours that B worked.

Since A could do the whole work in 14 hours, in 1 hour he would do $\frac{1}{14}$ of it; hence

$\frac{1}{14}x =$ the part of the work done by A in x hours.

Since B could do the whole work in 6 hours, in 1 hour he would do $\frac{1}{6}$ of it; hence

$\frac{1}{6}(10 - x) =$ the part of the work done by B in $10 - x$ hours.

But A and B together did the whole work; hence

$$\frac{1}{14}x + \frac{1}{6}(10 - x) = 1. \qquad (1)$$

From (1) $x = 7. \;\therefore\; 10 - x = 3.$

Hence A worked 7 hours, and B worked 3.

Example 6. Find the time between 5 and 6 o'clock at which the hands of a watch are together.

Suppose that the hands are together at x minutes after 5 o'clock.

At 5 o'clock the hour-hand is 25 minute-spaces ahead of the minute-hand; hence, while the minute-hand moves through x minute-spaces, the hour-hand will move through $x - 25$ such spaces. But the minute-hand moves 12 times as fast as the hour-hand; that is, in any given time the minute-hand passes over 12 times as many minute-spaces as the hour-hand. Hence

$$x = 12(x - 25). \qquad (1)$$

From (1) $x = 27\frac{3}{11}$.

Hence the hands are together at $27\frac{3}{11}$ minutes past 5 o'clock.

Exercise 21.

1. Find two numbers whose sum is 72, and whose difference is 8. *Ans.* 40 and 32.

2. Divide 25 into two parts whose difference is 5.

3. Divide 12 into two parts whose difference is 16.
Ans. 14 and -2.

4. The difference between two numbers is 8; if 2 be added to the greater, the result will be 3 times the smaller. Find the numbers.

5. A man walks 12 miles, then travels a certain distance by train, and then twice as far by coach as by train. If the whole journey is 78 miles, how far does he travel by train?

6. Find two numbers whose difference is 12, and whose sum is equal to $\frac{1}{2}$ their difference.

7. Find a number such that the sum of its sixth and ninth parts may be equal to 15.

8. Find the number of which the eighth, sixth, and fourth parts together make up -13. *Ans.* -24.

9. Find a number such that $\frac{4}{5}$ of it shall exceed $\frac{6}{7}$ of it by 2. *Ans.* -35.

10. Two numbers differ by 28, and one is $\frac{8}{9}$ of the other. Find them.

11. A, B, and C have a certain sum of money between them. A has $\frac{1}{2}$ of the whole, B has $\frac{1}{3}$ of the whole, and C has $50. How much have A and B?

12. A and B together have $75, and A has $5 more than B. How much has each?

13. A has $5 more than B, B has $20 more than C, and they have $360 between them. How much has each?

14. A has $15 more than B, B has $5 less than C, and they have $65 between them. How much has each?

15. A has \$100, and B has \$20. How much must A give B in order that B may have half as much as A?

16. The sum of two numbers is 38, and one of them exceeds twice the other by 2. What are the numbers?

17. Find a number which when multiplied by 8 exceeds 27 as much as 27 exceeds the original number.

18. Find two numbers of which the sum is 31, and which are such that one of them is less by 2 than half the other.

19. Divide 100 into two parts such that twice one part is equal to 3 times the other.

20. Four times the difference between the fourth and fifth parts of a certain number exceeds by 4 the difference between the third and seventh parts. Find the number.

21. Fifty times the difference between the seventh and eighth parts of a certain number exceeds half the number by 44. Find the number.

22. A father is 4 times as old as his son; in 24 years he will be only twice as old. Find their ages.

23. A is 25 years older than B, and A's age is as much above 20 as B's is below 85. Find their ages.

24. A's age is 6 times B's, and 15 years hence A will be 3 times as old as B. Find their ages.

25. Find a number such that if 5, 75, and 35 are added to it, the product of the first and third sums will be equal to the second sum multiplied by the number.

26. The difference between the squares of two consecutive whole numbers is 121. Find the numbers.

27. Divide \$380 between A, B, and C, so that B may have \$30 more than A, and C may have \$20 more than B.

28. The sum of the ages of A and B is 30 years, and 5 years hence A will be 3 times as old as B. Find their present ages.

29. The length of a room exceeds its breadth by 3 feet; if the length had been increased by 3 feet, and the breadth diminished by 2 feet, the area would not have been altered. Find the length and breadth of the room.

30. The length of a room exceeds its breadth by 8 feet; if each had been increased by 2 feet, the area would have been increased by 60 square feet. Find the dimensions of the room.

31. The width of a room is $\frac{2}{3}$ of its length. If the width had been 3 feet more, and the length 3 feet less, the room would have been square. Find the dimensions of the room.

32. A, B, and C have $1285 between them; A's share is greater than $\frac{5}{6}$ of B's by $25, and C's is $\frac{4}{15}$ of B's. Find the share of each.

33. If silk costs 6 times as much as linen, and I spend $66 in buying 40 yards of silk and 24 yards of linen, find the cost of each per yard.

34. If $600 be divided among 10 men, 20 women, and 40 children, so that each man receives $15 more than each child, and each woman receives as much as two children, find what each receives.

35. Divide $152 among 5 men, 7 women, and 30 children, giving to each man $4 more than to each woman, and to each woman 3 times as much as to each child.

36. A sum of money is divided between three persons, A, B, and C, so that A and B have $60 between them, A and C have $65, and B and C have $75. How much has each?

37. A dealer bought four horses for $1150; the second cost him $60 more than the first, the third $30 more than the second, and the fourth $10 more than the third. How much did each cost?

38. Two coaches start at the same time from York and London, a distance of 200 miles, travelling one at 9½ miles an hour, the other at 9¼ miles an hour. In how many hours after starting did they meet, and how far from London? *Ans.* 10⅔ hours; 98⅔ miles from London.

39. A man leaves ½ of his property to his wife, ⅓ to his son, and the remainder, which is $2500, to his daughter. How much did he leave to his wife and son each?

Let $x =$ the *number* of dollars which he left in all.

40. A man divided his property between his three children so that the eldest received twice as much as the second, and the second twice as much as the youngest. The eldest received $3750 more than the youngest. How much did each receive?

41. A third of the length of a post is in the mud, a fourth is in the water, and 5 feet is above the water. Find the length of the post.

42. A flock of sheep and goats together number 84. There are 3 goats to every 4 sheep. How many are there of each?

43. Find the time between 3 and 4 at which the hands of a clock are together.

44. A can do a piece of work in 30 days which B can do in 20 days. A begins the work, but after a time B takes his place, and the whole work is finished in 25 days from the beginning. How long did A work?

45. A can do a piece of work in 20 days which B can do in 30 days. A begins work, but after a time B takes his place and finishes it. B worked 10 days longer than A. How long did A work?

46. One number exceeds another by 3, while its square exceeds the square of the second by 99. Find the numbers.

47. Of two consecutive numbers, $\frac{1}{5}$ of the greater exceeds $\frac{1}{7}$ of the less by 3. Find the numbers.

48. A garrison of 1000 men having provisions for 60 days was reinforced after 10 days, and from that time the provisions lasted only 20 days. Find the number in the reinforcement.

49. In a mixture of spirits and water half of the whole plus 25 gallons was spirits; and a third of the whole minus 5 gallons was water. How many gallons were there of each?

50. At 3 o'clock, A starts upon a journey at the rate of 4 miles an hour, and after 15 minutes B starts from the same place, and follows A at the rate of $4\frac{3}{4}$ miles an hour. When did B overtake A?

51. A fish was caught whose tail weighed 9 pounds; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail. What did the fish weigh?

52. Find a number such that if $\frac{3}{8}$ of it be subtracted from 20, and $\frac{5}{11}$ of the remainder from $\frac{1}{4}$ of the original number, 12 times the second remainder shall be half the original number.

53. A cistern can be filled in half an hour by a pipe A, and emptied in 20 minutes by another pipe B; after A has been opened 20 minutes, B is also opened for

12 minutes, when A is closed, and B remains open for 5 minutes more, after which there are 13 gallons in the cistern. What was the capacity of the cistern?

54. A father was 24 years old when his eldest son was born; and if both live till the father is twice as old as he now is, the son will then be 8 times as old as now. Find the father's present age.

55. A person being asked how many ducks and geese he had in his yard, said: "If I had 8 more of each, I should have 8 ducks for 7 geese; and if I had 8 less of each, I should have 7 ducks for 6 geese." How many had he of each?

56. If 19 lbs. of gold weigh 18 lbs. in water, and 10 lbs. of silver weigh 9 lbs. in water, find the quantities of gold and silver in a mass of gold and silver weighing 106 lbs. in air, and 99 lbs. in water.

57. The sum of $1650 is laid out in two investments, by one of which 15 per cent is gained, and by the other 8 per cent is lost; and the amount of the returns is $1725. Find the amount of each investment.

CHAPTER VIII.

INVOLUTION.

106. Involution is the operation of raising a number to any required *power*.

If m and n are both positive integers, we have the following Laws of Exponents for involution:

107. *The* n*th power of the* m*th power of any base equals the* mn*th power of that base;* that is,

$$(a^m)^n = a^{mn}.$$

For $$(a^m)^n = a^m \cdot a^m \cdots \text{ to } n \text{ factors}$$
$$= a^{m+m+\cdots \text{ to } n \text{ terms}} \qquad \S\ 55$$
$$= a^{mn}.$$

Thus $$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^{2\times 3} = a^6.$$

108. *The* n*th power of the product of any number of factors equals the product of the* n*th powers of those factors;* that is,

$$(ab)^n = a^n b^n.$$

For $$(ab)^n = (ab)(ab) \cdots \text{ to } n \text{ factors}$$
$$= (aa \cdots \text{ to } n \text{ factors})(bb \cdots \text{ to } n \text{ factors})$$
$$= a^n b^n.$$

In like manner the law can be proved for any number of factors.

Thus $$(-5x^2y^3)^3 = (-5x^2y^3)(-5x^2y^3)(-5x^2y^3)$$
$$= (-5)^3(x^2)^3(y^3)^3$$
$$= -125x^6y^9. \qquad \S\ 107$$

109. *The* n*th power of the quotient of any two numbers equals the quotient of their* n*th powers;* that is,

$$\left(\frac{a}{x}\right)^n=\frac{a^n}{x^n}.$$

For $$\left(\frac{a}{x}\right)^n=\frac{a}{x}\cdot\frac{a}{x}\cdot\frac{a}{x}\cdots\text{to } n \text{ factors}$$

$$=\frac{a^n}{x^n}. \qquad \S\ 75$$

Thus $$\left(-\frac{2x^2y^3}{3a^3b^2c}\right)^3=(-1)^3\left(\frac{2x^2y^3}{3a^3b^2c}\right)^3 \qquad \S\ 108$$

$$=-\frac{(2x^2y^3)^3}{(3a^3b^2c)^3} \qquad \S\ 109$$

$$=-\frac{8x^6y^9}{27a^9b^6c^3}. \qquad \S\S\ 107,\ 108$$

110. Quality of a Power. From § 50, it follows that

Any odd power of a negative base will be negative; all other powers will be positive.

Exercise 22.

Find the value of

1. $(x^3)^4$.
2. $(x^4)^3$.
3. $(ax^3)^2$.
4. $(-a^2x)^2$.
5. $(-by^2)^3$.
6. $(-2ax^3)^2$.
7. $(-a^2x)^5$.
8. $(ab^3x^2)^4$.
9. $(-x^2yz^3)^6$.
10. $(-2ab^2x^3)^4$.
11. $(-2a^2bc^3)^5$.
12. $(-5a^2x^3y^2)^3$.
13. $\left(-\frac{2ax^2}{3b^2y}\right)^2$.
14. $\left(-\frac{a^2xy^2}{bc^2z^3}\right)^3$.
15. $\left(-\frac{2xyz^2}{3a^2bc^3}\right)^4$.
16. $\left(\frac{2axy^3}{b^2cz}\right)^5$.
17. $\left(-\frac{abc^2}{x^2yz^3}\right)^6$.
18. $\left(-\frac{ax^2y}{bc^2z}\right)^7$.
19. $(-x^ry^s)^{2n}$.

111. The Square of any Binomial *is equal to the sum of the squares of its terms plus twice their product.*

For let a and b denote any two terms; then $a+b$ will denote any binomial, and

$$\begin{aligned}(a+b)^2 &\equiv (a+b)(a+b)\\ &\equiv (a+b)a+(a+b)b \qquad \text{§ 67}\\ &\equiv a^2+2ab+b^2,\end{aligned}$$

which proves the theorem.

Thus $(3x+5y)^2=(3x)^2+(5y)^2+2(3x)(5y)$
$= 9x^2+25y^2+30xy.$

Also, $(2x-3y)^2=(2x)^2+(-3y)^2+2(2x)(-3y)$
$=4x^2+9y^2-12xy.$

112. The Square of any Polynomial *is equal to the sum of the squares of its several terms plus twice the product of each term into each of the terms that follow it.*

For, if in the identity

$$(a+x)^2=a^2+x^2+2ax \qquad (1)$$

we put $b+y$ for x, we shall obtain

$$\begin{aligned}(a+b+y)^2 &= a^2+(b+y)^2+2a(b+y)\\ &= a^2+b^2+y^2+2ab+2ay+2by. \qquad (2)\end{aligned}$$

Hence the theorem holds true for a trinomial.

In like manner, putting $c+d$ for y in (2), we may prove the theorem for a polynomial of four terms; and so on.

Thus $(x^2+2y-3c)^2$

$$\begin{aligned}&=(x^2)^2+(2y)^2+(-3c)^2+2(x^2)(2y)+2(x^2)(-3c)\\ &\quad +2(2y)(-3c)\\ &=x^4+4y^2+9c^2+4x^2y-6cx^2-12cy.\end{aligned}$$

Also, $(2x - a + c^2 - 3y)^2$

$$= 4x^2 + a^2 + c^4 + 9y^2 - 4ax + 4c^2x - 12xy - 2ac^2 + 6ay - 6c^2y.$$

Exercise 23.

Write out the square of

1. $2a + x.$
2. $x^2 + y^2.$
3. $3a - 5b.$
4. $3a^3 - b^3x.$
5. $-2a^2 + 5b^2.$
6. $-ax^2 + by^2.$
7. $3\,abc - 4x^2y.$
8. $-2z^2 - abx^2.$
9. $-4a^2 + 3cz^3.$
10. $a + b + c.$
11. $a + b - c.$
12. $a - b + c.$
13. $a - b - c.$
14. $a + 2b + 4.$
15. $x + 2y + 3z.$
16. $2 + 2x - 3x^2.$
17. $2x^2 - 3x - 2.$
18. $a^3 - 2b^3 + 3c^3.$
19. $3x^2 - 6x - 6.$
20. $x + y + z + v.$
21. $x + y - z - v.$
22. $x - y - z - v.$
23. $x^3 + x^2 - 2x - 2.$
24. $a + 2b - 3c + 4d.$
25. $1 + x - x^2 + x^3.$
26. $3a^2 - x^2 + c^2 - 2y^2.$
27. $4x^2 - 3a - 4c - 3y^2.$
28. $3x^3 - 2a^2 + 4b - y^3.$

113. The Binomial Theorem. Let a and b denote any two terms; then $a + b$ will denote any binomial, and by § 111 we shall have

$$(a + b)^2 = a^2 + 2ab + b^2. \qquad (1)$$

Multiplying both members of (1) by $a + b$, we obtain

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3. \qquad (2)$$

Multiplying both members of (2) by $a + b$, we obtain

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4. \qquad (3)$$

Multiplying (3) by $a+b$, we obtain

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

The second member of each of these identities is called the **Expansion** of its first member. By inspecting the terms of the expansion of each of these powers of $a+b$, we discover the following laws of exponents and coefficients:

(i.) *The exponent of* a *in the first term of each expansion equals the exponent of the binomial, and it decreases by unity in each succeeding term.*

(ii.) *The exponent of* b *is unity in the second term of the expansion, and it increases by unity in each succeeding term.*

(iii.) *The coefficient of the first term of each expansion is unity, and that of the second term equals the exponent of the binomial.*

(iv.) *If in any term the coefficient be multiplied by* a's *exponent, and this product be divided by* b's *exponent plus* 1, *the result will be the coefficient of the next term.*

For example let us consider the expansion of $(a+b)^5$. By (iv.), from the second term, $5a^4b$, we obtain $5 \times 4 \div 2$, or 10, which is the coefficient of the next term. From the third term, $10a^3b^2$, we obtain $10 \times 3 \div 3$, or 10, which is the coefficient of the next term; and so on.

By the laws given in (i.), (ii.), (iii.), and (iv.) any binomial with *any exponent* may be expanded in a series. These laws are called the *Binomial Theorem.*

NOTE. The general demonstration of this theorem was first given by Sir Isaac Newton. It was considered one of the finest of his discoveries, and the theorem was engraved on his tomb. Its statement in symbols will be found in § 240.

The extension of the proof here given to the sixth, seventh, etc., powers is left as an exercise for the student.

In the expansion of $(a+b)^4$ there are 5 terms, each of the fourth degree in a and b, and the first two coefficients are repeated in inverse order after the third term. In the expansion of $(a+b)^5$ there are 6 terms, each of the fifth degree in a and b, and the first three coefficients are repeated in inverse order after the third term. Hence we note that

The sum of the exponents of a and b in any term of the expansion is equal to the exponent of the binomial.

The number of terms in the expansion is one greater than the exponent of the binomial.

The coefficients are repeated in the inverse order after passing the middle term or half the number of terms, so that the coefficients of the last half of the expansion can be written out from the first half.

NOTE. The last two statements hold true only when the exponent of the binomial is a positive integer.

EXAMPLE 1. Expand $(x+y)^7$.

Here x takes the place of a, and y that of b.

$$\therefore\ (x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.$$

According to (i.), (ii.), and (iii.), the first and second terms are x^7 and $7x^6y$ respectively. By (iv.), the coefficient of the third term is $7 \times 6 \div 2$, or 21; that of the fourth term is $21 \times 5 \div 3$, or 35. The coefficients of the last four terms are those of the first four in reverse order.

EXAMPLE 2. Expand $(x-y)^5$, or $[x+(-y)]^5$.

Here x takes the place of a, and $-y$ that of b.

$$\begin{aligned}\therefore\ (x-y)^5 &= x^5 + 5x^4(-y) + 10x^3(-y)^2 + 10x^2(-y)^3 \\ &\quad + 5x(-y)^4 + (-y)^5 \\ &= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5.\end{aligned}$$

In the expansion of $(x-y)^5$, the sign before the second, the fourth, and the sixth term is $-$; that is, the sign before each even term is $-$. The reason for this is that each even term contains an odd power of -1. The expansions of $(x-y)^5$ and $(x+y)^5$ differ only in the signs before the even terms.

Exercise 24.

Expand by the binomial theorem

1. $(x+y)^6$.
2. $(x+y)^8$.
3. $(x+y)^9$.
4. $(x+y)^{10}$.
5. $(x-y)^3$.
6. $(x-y)^4$.
7. $(x-y)^6$.
8. $(c-d)^7$.
9. $(x+1)^3$.
10. $(1+y)^4$.
11. $(a-1)^6$.
12. $(2x^2-y^3)^4$.

Here $2x^2$ takes the place of a, and $-y^3$ that of b.

$$\begin{aligned}\therefore\ (2x^2-y^3)^4 &= (2x^2)^4+4(2x^2)^3(-y^3)+6(2x^2)^2(-y^3)^2\\ &\quad +4(2x^2)(-y^3)^3+(-y^3)^4\\ &= 16x^8-32x^6y^3+24x^4y^6-8x^2y^9+y^{12}.\end{aligned}$$

13. $(2a+3)^4$.
14. $(3a+2b)^5$.
15. $(3x-2y)^4$.
16. $(2x^2-a)^5$.
17. $(2a^2-c^3)^6$.
18. $(x^2-1)^7$.
19. $(2+\frac{1}{2}x^2)^4$.
20. $(2a-\frac{1}{2}x)^5$.
21. $\left(\frac{1}{a^2}-\frac{1}{b^2}\right)^3$.
22. $\left(\frac{1}{x}-\frac{1}{y}\right)^5$.
23. $(z-3x)^7$.
24. $(2a-c+x^2)^3$.

Since any polynomial can be written as a binomial, the binomial theorem may be used to expand any power of a polynomial.

Thus we may write,

$$\begin{aligned}(2a-c+x^2)^3 &= [(2a-c)+x^2]^3\\ &= (2a-c)^3+3(2a-c)^2(x^2)+3(2a-c)(x^2)^2\\ &\quad +(x^2)^3\\ &= 8a^3-12a^2c+6ac^2-c^3+12a^2x^2-12acx^2\\ &\quad +3c^2x^2+6ax^4-3cx^4+x^6.\end{aligned}$$

25. $(x^2+x+1)^3$.
26. $(x^2-x+2)^3$.
27. $(3x^2-5x+1)^3$.
28. $(2x-3a+b)^3$.

CHAPTER IX.

EVOLUTION.

114. The *n*th **Root** of a number or expression is one of its n equal factors. Thus the *square root* of a number is one of its two equal factors; the *cube root* of a number is one of its three equal factors; and so on. A root is indicated by the sign $\sqrt{\ }$, called the **Radical Sign.** In the expression $\sqrt[n]{a}$, n denotes the number of equal factors into which a is to be resolved, and is called the **Index** of the root. If no index is written, 2 is understood.

Thus $\sqrt{a}$ is read, 'the second,' or 'the square root, of a'; $\sqrt[3]{a}$ is read, 'the third,' or 'the cube, root of a'; and $\sqrt[n]{a}$ is read 'the nth root of a.'

$$\sqrt{4} = \pm 2, \quad \because \quad (\pm 2)^2 = 4; \qquad \sqrt[4]{81} = \pm 3, \quad \because \quad (\pm 3)^4 = 81;$$
$$\sqrt[3]{-64} = -4, \quad \because \quad (-4)^3 = -64; \quad \sqrt[5]{32} = 2, \qquad \because \qquad 2^5 = 32.$$

115. A root is said to be **Even** or **Odd** according as its index is even or odd.

Thus $\sqrt[4]{a}$ denotes an even, and $\sqrt[5]{a}$ denotes an odd, root of a.

116. Real Numbers. The positive and negative numbers defined in § 36 are called *Real Numbers.*

117. Odd Roots. *Any real number has one odd root of the same quality as the number itself.*

For any base has the same quality as any odd power of that base.

Thus $\sqrt[3]{8} = 2$, $\sqrt[3]{-27} = -3$, $\sqrt[5]{-32} = -2$, $\sqrt[5]{243} = 3$.

118. Even Roots. *Any positive number has two real even roots that are numerically equal, but opposite in quality.*

For if two numbers are arithmetically equal, but opposite in quality, their like even powers are equal.

Thus $\sqrt{81} = \pm 9$, $\because (\pm 9)^2 = 81$; $\sqrt[4]{81} = \pm 3$, $\because (\pm 3)^4 = 81$.

119. *A negative number has no even root that is real.*

For no *even* power of any *real* number can be a negative number.

120. Imaginary Numbers. The indicated even root of a negative number is called an *Imaginary Number.*

Thus the expressions $\sqrt{-3}$, $\sqrt[4]{-5}$, and $\sqrt[6]{-4}$ denote imaginary numbers. The symbol $\sqrt{-3}$ stands for that number the square of which equals -3; that is, $(\sqrt{-3})^2 = -3$.

NOTE. As the general solution of linear equations leads to the idea of real algebraic numbers, so the general solution of quadratic and higher equations leads to the idea of imaginary numbers.

By the solution of pure quadratic and higher equations it is shown that any real number has two square roots, three cube roots, four fourth roots, five fifth roots, and so on. Some of these roots are real numbers, others imaginary.

In this chapter we shall consider only the real odd root or the positive even root of any number or expression.

121. Axioms. (i.) *Like powers of equals are equal.*

(ii.) *Like roots of equals are equal.*

122. Evolution is the operation of finding any required root of a number or expression.

For evolution we have the following principles:

123. *The exponent of any base in the root is equal to the exponent of that base in the power divided by the index of the root.*

By § 107, $a^{mn} = (a^m)^n.$

Hence $\sqrt[n]{a^{mn}} = a^m$, or $a^{mn \div n}$. § 121

Thus $\sqrt[3]{a^6} = a^{6 \div 3} = a^2$, $\sqrt[5]{a^{15}} = a^{15 \div 5} = a^3$.

124. *The* nth *root of a product is equal to the product of the* nth *roots of its factors.*

By § 108, $ab = (\sqrt[n]{a} \cdot \sqrt[n]{b})^n.$

$\therefore \sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$ § 121

Thus $\sqrt[5]{-32\,a^{10}} = \sqrt[5]{-32} \cdot \sqrt[5]{a^{10}} = -2\,a^2$.

125. *The* nth *root of the quotient of two numbers is equal to the quotient of their* nth *roots.*

By § 109,

$$\frac{a}{b} = \left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n.$$

$$\therefore \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Thus

$$\sqrt[3]{\frac{-125\,x^6}{216\,a^{12}}} = \frac{\sqrt[3]{-125\,x^6}}{\sqrt[3]{216\,a^{12}}}$$
$$= (-5\,x^2) \div (6\,a^4).$$

Example 1. Find the value of $\sqrt[3]{-a^9b^6}$.

$$\sqrt[3]{-a^9b^6} = \sqrt[3]{-1} \cdot \sqrt[3]{a^9} \cdot \sqrt[3]{b^6} \qquad \text{§ 124}$$
$$= -a^3b^2. \qquad \text{§ 123}$$

Example 2. Find the value of $\sqrt[5]{-\frac{32\,x^5y^{10}}{a^5b^{10}z^{15}}}$.

$$\sqrt[5]{-\frac{32\,x^5y^{10}}{a^5b^{10}z^{15}}} = \sqrt[5]{-1} \cdot \sqrt[5]{\frac{32\,x^5y^{10}}{a^5b^{10}z^{15}}} \qquad \text{§ 124}$$
$$= -1\,\frac{\sqrt[5]{32\,x^5y^{10}}}{\sqrt[5]{a^5b^{10}z^{15}}} \qquad \text{§ 125}$$
$$= -\frac{2\,xy^2}{ab^2z^3}. \qquad \text{§§ 124, 123}$$

126. A **Perfect** *n***th Power** is any expression of which the *n*th root is a *rational* expression.

Thus $25a^4b^2$ is a *perfect square;* for $\sqrt{25a^4b^2} = 5a^2b$.

Again, $a^2 + 2ab + b^2$ is a *perfect square;*

for $\sqrt{a^2 + 2ab + b^2} = a + b$.

Likewise, $-8x^3y^6$ is a *perfect cube;* for $\sqrt[3]{-8x^3y^6} = -2xy^2$.

127. Surds. If any required root of an expression cannot be obtained exactly, its indicated root is called a *Surd.*

Thus $\sqrt{5}$, $\sqrt{a}$, $\sqrt[3]{7}$, and $\sqrt{a-x}$ are surds.

An **Irrational** or **Surd Expression** is one that contains one or more surds.

Thus $\sqrt{a} + \sqrt{b}$ and $a + c\sqrt{2}$ are surd expressions.

EXAMPLE. Find the value of $\sqrt{\frac{4}{9} + \frac{16}{3}}$.

$$\begin{aligned}\sqrt{\tfrac{4}{9} + \tfrac{16}{3}} &= \sqrt{\tfrac{52}{9}} \\ &= \sqrt{\tfrac{4}{9} \cdot 13} \\ &= \sqrt{\tfrac{4}{9}} \cdot \sqrt{13} = \tfrac{2}{3}\sqrt{13}. \qquad \S\ 124\end{aligned}$$

Since $\frac{4}{9}$ is the greatest perfect square in $\frac{52}{9}$, we resolve $\frac{52}{9}$ into the two factors $\frac{4}{9}$ and 13.

Exercise 25.

Find the value of

1. $\sqrt{4a^2b^2}$.
2. $\sqrt{9x^6y^2}$.
3. $\sqrt{25a^4b^6}$.
4. $\sqrt{100a^8}$.
5. $\sqrt[3]{27a^6b^3}$.
6. $\sqrt[3]{-8x^9y^6}$.
7. $\sqrt[3]{-a^9x^6y^{12}}$.
8. $\sqrt[3]{-64x^3y^6}$.
9. $\sqrt[3]{343a^{12}b^6}$.
10. $\sqrt[4]{81a^8b^4}$.
11. $\sqrt[4]{16x^{12}y^8}$.
12. $\sqrt[5]{32x^5y^{10}}$.

13. $\sqrt{\frac{a^6b^4}{16}}$. 17. $\sqrt[5]{\frac{32x^5y^{10}}{-a^{10}b^{15}}}$. 21. $\sqrt{\frac{4}{5}+\frac{12}{25}}$.

14. $\sqrt{\frac{400a^{10}b^6}{81x^8y^{12}}}$. 18. $\sqrt[7]{-\frac{128}{a^{42}b^{63}}}$. 22. $\sqrt{\frac{7}{8}-\frac{7}{64}}$.

15. $\sqrt[3]{\frac{125a^3b^6}{216x^9y^6}}$. 19. $\sqrt{\frac{2}{3}+\frac{5}{9}}$. 23. $\sqrt{\frac{5}{6}-\frac{1}{18}}$.

16. $\sqrt[3]{-\frac{27x^6y^3}{64a^9b^{12}}}$. 20. $\sqrt{\frac{3}{4}-\frac{11}{16}}$. 24. $\sqrt{\frac{6}{7}-\frac{2}{49}}$.

SQUARE ROOT OF POLYNOMIALS.

128. Trinomials. By § 111

$$(a \pm x)^2 = a^2 \pm 2ax + x^2.$$

Hence a trinomial is a perfect square, if when its terms are arranged according to the powers of some letter, its second term is plus or minus twice the product of the square roots of its first and last terms.

To obtain the square root of such a trinomial, *connect the square roots of its first and last terms by the sign before its second term.*

Thus $4a^2 + 12ab + 9b^2$ is a perfect square;

for $$2\sqrt{4a^2} \times \sqrt{9b^2} = 2 \times 2a \times 3b = 12ab.$$

$$\therefore \sqrt{4a^2 + 12ab + 9b^2} = 2a + 3b.$$

EXAMPLE 1. Find the square root of $9x^2 - 24x + 16$.

$$2\sqrt{9x^2} \times \sqrt{16} = 2 \times 3x \times 4 = 24x.$$

$$\therefore \sqrt{9x^2 - 24x + 16} = 3x - 4.$$

EXAMPLE 2. Find the square root of $36a^4 + b^4 - 12a^2b^2$.

$$2\sqrt{36a^4} \times \sqrt{b^4} = 2 \times 6a^2 \times b^2 = 12a^2b^2.$$

$$\therefore \sqrt{36a^4 - 12a^2b^2 + b^4} = 6a^2 - b^2.$$

Exercise 26.

Find the square root of

1. $a^2 + 2ab + b^2$.
2. $x^2 + 12x + 36$.
3. $x^2 + 9 + 6x$.
4. $x^2 + 121 - 22x$.
5. $a^2 + 49 - 14a$.
6. $a^2 + 25 - 10a$.
7. $1 - 8x + 16x^2$.
8. $4a^2 + 9b^2 - 12ab$.
9. $9a^4 + 24a^2b^2 + 16b^4$.
10. $x^2 + \frac{1}{4}y^2 + xy$.
11. $4a^2x^2 + 4abxy + b^2y^2$.
12. $9a^2 + 25b^2 - 30ab$.
13. $25a^4x^2 - 30a^2b^2x + 9b^4$.
14. $25a^4 + 25b^2 - 50a^2b$.
15. $9a^4 + 16b^4 - 24a^2b^2$.
16. $a^2 + 25b^6 - 10ab^3$.
17. $\frac{1}{9}x^2 + \frac{1}{4}y^2 + \frac{1}{3}xy$.
18. $\frac{4}{9}a^4 + \frac{16}{25}b^2 - \frac{16}{15}a^2b$.
19. $4 - \frac{4x}{y} + \frac{x^2}{y^2}$.
20. $\frac{x^2}{y^2} + \frac{10x}{y} + 25$.
21. $\frac{x^2}{y^2} - \frac{2ax}{by} + \frac{a^2}{b^2}$.
22. $\frac{x^2}{4y^2} - \frac{2x}{y} + 4$.
23. $\frac{64x^2}{9y^2} + \frac{32x}{3y} + 4$.
24. $\frac{9x^2}{25} - 2 + \frac{25}{9x^2}$.

129. Any polynomial that is a perfect square, and contains only two different powers of a particular letter, can be written as a trinomial, and its square root obtained by § 128.

Example 1. Find the square root of the expression

$$x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.$$

Arranging the expression according to the descending powers of x, we have

$$x^2 + 2x(y + z) + (y^2 + z^2 + 2yz),$$

or $$x^2 + 2x(y + z) + (y + z)^2.$$

$$\therefore \sqrt{x^2 + y^2 + z^2 + 2xy + 2xz + 2yz} = x + y + z.$$

Example 2. Find $\sqrt{a^4 + 4b^4 + 9c^4 + 4a^2b^2 - 6a^2c^2 - 12b^2c^2}$.

Arranging according to descending powers of a, we have

$$a^4 + 2a^2(2b^2 - 3c^2) + (4b^4 + 9c^4 - 12b^2c^2),$$

or $$a^4 + 2a^2(2b^2 - 3c^2) + (2b^2 - 3c^2)^2.$$

$$\therefore \sqrt{a^4 + 4b^4 + 9c^4 + 4a^2b^2 - 6a^2c^2 - 12b^2c^2} = a^2 + 2b^2 - 3c^2.$$

Example 3. Find square root of

$$a^6 - 2a^5 + 3a^4 + 2a^3(b-1) + a^2(1-2b) + 2ab + b^2.$$

This expression contains only b^2 and b; hence we arrange it according to the powers of b; we then have

$$b^2 + 2b(a^3 - a^2 + a) + (a^6 - 2a^5 + 3a^4 - 2a^3 + a^2).$$

This expression is a perfect square if its last term is the square of $a^3 - a^2 + a$. But by § 112 we have

$$(a^3 - a^2 + a)^2 = a^6 - 2a^5 + 3a^4 - 2a^3 + a^2.$$

Hence the given expression is equal to

$$b^2 + 2b(a^3 - a^2 + a) + (a^3 - a^2 + a)^2,$$

of which the square root is $b + a^3 - a^2 + a$.

Exercise 27.

Find the square root of

1. $c^2 - 6c(a+b) + 9(a+b)^2$.
2. $a^2 + b^2 + 4c^2 + 2ab + 4ac + 4bc$.
3. $4a^2 + b^2 + 9c^2 + 6bc - 12ca - 4ab$.
4. $4a^4 + b^4 + c^4 - 2b^2c^2 - 4a^2c^2 + 4a^2b^2$.
5. $x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz$.
6. $25a^4 + 9b^4 + 4c^4 - 12b^2c^2 + 20c^2a^2 - 30a^2b^2$.
7. $6acx^5 + 4b^2x^4 + a^2x^{10} + 9c^2 - 12bcx^2 - 4abx^7$.
8. $-6b^2c^2 + 9c^4 + b^4 - 12c^2a^2 + 4a^4 + 4a^2b^2$.
9. $6ab^2c - 4a^2bc + a^2b^2 + 4a^2c^2 + 9b^2c^2 - 12abc^2$.

Ans. $ab - 2ac + 3bc$.

130. To find the square root of any polynomial.

The rule is given by the identity,

$$(a+b)^2 \equiv a^2 + (2a+b)b. \qquad (1)$$

At first a represents the first term of the root, and b the second term. After the first two terms of the root have been found, a denotes the part of the root already found, and b the next term of the root. $2a$ denotes the *trial* divisor for obtaining b; and $2a+b$ the *true* divisor.

The work may be arranged as below, the root being placed at the right of the power.

$$\begin{array}{r|l|l}
 & a^2 + 2ab + b^2 & \underline{a+b} \\
 & \underline{a^2} & \\
2a+b & 2ab + b^2 & \\
 & \underline{2ab + b^2} &
\end{array}$$

Now a, the first term of the root, is the square root of a^2, the first term of the given square. Subtract a^2 from the given square, and the remainder is $2ab + b^2$. Hence b, the next term of the root, is obtained by dividing $2ab$, the first term of the remainder, by $2a$, that is, by *twice the part of the root already found.*

Again, if to $2a$, the *trial* divisor, we add b, the next term of the root, we obtain the *true* divisor $2a + b$.

This true divisor multiplied by b gives $2ab + b^2$, which with a^2 makes up the square of $a + b$.

Example 1. Find the square root of $9x^2 - 42xy + 49y^2$.

$$\begin{array}{rl|l}
 & 9x^2 - 42xy + 49y^2 & \underline{3x - 7y}, \text{ Root.} \\
a^2 = & \underline{9x^2} & \\
2a + b = 6x - 7y & -42xy + 49y^2 & \\
(2a+b)b = & \underline{-42xy + 49y^2} &
\end{array}$$

The expression is arranged according to descending powers of x. Here a, the first term of the root, is $\sqrt{9x^2}$, or $3x$.

Subtract a^2, or $9x^2$, and the remainder is $-42xy + 49y^2$. Now $2a$, the *trial* divisor, is $6x$; hence b, the second term of the root, is $(-42xy) \div (6x)$, or $-7y$, which is written as the next term both of the root and the divisor.

$$\therefore\ 2a + b = 6x - 7y, \text{ the } \textit{true} \text{ divisor.}$$

Multiply this by b, or $-7y$, and subtract the product.

Now from the given expression we have subtracted a^2 and $(2a+b)b$, or the two parts of $(a+b)^2$; that is, we have subtracted $(3x-7y)^2$, since $a = 3x$, and $b = -7y$.

Hence, as there is no remainder, the given expression is equal to $(3x-7y)^2$, and its square root is $3x-7y$.

Example 2. Find the square root of

$$25a^2x^2 - 12a^3x + 16x^4 + 4a^4 - 24ax^3.$$

Formula, $\qquad (a+b)^2 = a^2 + (2a+b)b.$

Arrange according to descending powers of x.

$$
\begin{array}{r|l|l}
 & 16x^4 - 24ax^3 + 25a^2x^2 - 12a^3x + 4a^4 & \underline{4x^2 - 3ax + 2a^2} \\
 & \underline{16x^4} & \\
8x^2 - 3ax & -24ax^3 + 25a^2x^2 & \\
 & \underline{-24ax^3 + \ \ 9a^2x^2} & \\
8x^2 - 6ax + 2a^2 & 16a^2x^2 - 12a^3x + 4a^4 & \\
 & \underline{16a^2x^2 - 12a^3x + 4a^4} &
\end{array}
$$

Here a, the first term of the root, is $4x^2$. Subtract a^2, or $16x^4$. Now $2a$, the *trial* divisor, is $8x^2$; hence b, the second term of the root, is $(-24ax^3) \div (8x^2)$, or $-3ax$.

$$\therefore\ 2a + b = 8x^2 - 3ax, \text{ the } \textit{true} \text{ divisor.}$$

Multiply this by b, or $-3ax$, and subtract the product; then in all we have subtracted $a^2 + (2a+b)b$, or $(a+b)^2$; that is, we have subtracted $(4x^2 - 3ax)^2$, since $a = 4x^2$, and $b = -3ax$.

Next let $a = 4x^2 - 3ax$, *the part of the root already found*, of which the square has been subtracted. Then $2a$, the *trial* divisor, is $8x^2 - 6ax$; and b, the next term of the root, is $(16a^2x^2) \div (8x^2)$, or $2a^2$.

$$\therefore\ 2a + b = 8x^2 - 6ax + 2a^2, \text{ the } \textit{true} \text{ divisor.}$$

Multiply this by b, or $2a^2$, and subtract the product; then in all we have subtracted $(a+b)^2$, that is, $(4x^2 - 3ax + 2a^2)^2$, since $a = 4x^2 - 3ax$, and $b = 2a^2$.

Hence, as there is no remainder, the given expression is equal to $(4x^2 - 3ax + 2a^2)^2$, and its square root is $4x^2 - 3ax + 2a^2$.

By the process described above, we may extract the square root of any polynomial. In applying it, the learner should remember

(i.) To arrange the given expression and each remainder according to the ascending or descending powers of the same letter.

(ii.) That, at each stage of the process, the trial divisor is twice the part of the root already found, and

(iii.) That the true divisor is obtained by adding to the trial divisor the next term of the root.

An equality that furnishes a rule for performing an operation or for solving a problem is called a **Formula.**

Thus identity (1) is a formula.

Exercise 28.

Find the square root of

1. $x^4 + 2x^3 + 3x^2 + 2x + 1$.
2. $4x^4 - 8x^3 + 4x + 1$.
3. $9x^4 - 36x^3 + 72x + 36$.
4. $4x^4 + 4x^3 - \frac{1}{2}x + \frac{1}{16}$.
5. $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$.
6. $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.
7. $16 - 96x + 216x^2 - 216x^3 + 81x^4$.
8. $1 + 4x + 10x^2 + 12x^3 + 9x^4$.
9. $4x^4 - 4x^3 + 3x^2 - x + \frac{1}{4}$.
10. $1 - xy - \frac{15}{4}x^2y^2 + 2x^3y^3 + 4x^4y^4$.
11. $x^6 - 4x^5 + 6x^4 - 8x^3 + 9x^2 - 4x + 4$.
12. $9x^6 - 12x^5 + 22x^4 + x^2 + 12x + 4$.
13. $x^6 - 22x^4 + 34x^3 + 121x^2 - 374x + 289$.
14. $a^2 - ax + \frac{1}{4}x^2 + 8a - 4x + 16$.

15. $x^8+2x^7+x^6-4x^5-12x^4-8x^3+4x^2+16x+16.$

16. $(1+2x^2)^2-4x(1-x)(1+2x).$

17. $x^4+2x^3(y+z)+x^2(y^2+z^2+4yz)+2xyz(y+z)+y^2z^2.$

18. $x^4-2x+\frac{1}{9}+\frac{29}{3}x^2-6x^3.$

19. $-3a^3+\frac{25}{9}+a^4-5a+\frac{67}{12}a^2.$

20. $\frac{1}{4}x^4+4x^2+\frac{1}{3}ax^2+\frac{1}{9}a^2-2x^3-\frac{4}{3}ax.$

21. $24+\dfrac{16y^2}{x^2}-\dfrac{8x}{y}+\dfrac{x^2}{y^2}-\dfrac{32y}{x}.$

In the polynomial

$$x^3+x^2+x+1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3} \qquad (1)$$

each term after the first is obtained by dividing the preceding term by x; hence we regard all the terms in expression (1) as arranged according to the powers of x.

Arranging the given expression according to the powers of x, we have

$$\frac{x^2}{y^2}-\frac{8x}{y}+24-\frac{32y}{x}+\frac{16y^2}{x^2}.$$

22. $\dfrac{a^4}{4}+\dfrac{a^3}{x}+\dfrac{a^2}{x^2}-ax-2+\dfrac{x^2}{a^2}.$

23. $\dfrac{9a^2}{x^2}-\dfrac{6a}{5x}+\dfrac{101}{25}-\dfrac{4x}{15a}+\dfrac{4x^2}{9a^2}.$

24. $4x^4+32x^2+96+\dfrac{64}{x^4}+\dfrac{128}{x^2}.$

SQUARE ROOT OF ARITHMETICAL NUMBERS.

131. *If a number contains* n *integral figures, its square root will contain* $\frac{1}{2}$n *integral figures if* n *is even, and* $\frac{1}{2}$(n + 1) *if* n *is odd.*

For $\sqrt{1}=1$, and $\sqrt{100}=10$; hence the square root of any number between 1 and 100 lies between 1 and 10; that is, if a number contains one or two integral figures, its square root will contain one integral figure. Again, $\sqrt{100} = 10$; and $\sqrt{10000} = 100$; hence the square root of any number between 100 and 10000 lies between 10 and 100; that is, if a number contains three or four integral figures, its square root will contain two; and so on.

Thus the square root of 347254 will contain three integral figures, that is, hundreds, tens, and units; and the square root of 2174321 will contain four integral figures.

132. The first step in finding the square root of an arithmetical number, is to divide its integral figures into groups of two figures each, beginning at units' place. The last group may have only one figure.

We thus determine the number of integral figures in the root, and indicate the part of the number from which each figure of the root is to be obtained.

Thus in the square root of 53824 there are hundreds, tens, and units; and the hundreds' figure will be the square root of the greatest square in 5.

When the figures of a number have been divided into groups of two figures each, the rule for finding its square root is given by the formula,

$$(a+b)^2 \equiv a^2 + (2a+b)b,$$

in which *a* denotes tens with reference to *b*; that is, when *a* denotes hundreds, *b* denotes tens, and when *a* denotes tens, *b* denotes units.

EXAMPLE. Find the square root of 53824.

$$\begin{array}{rr|l}
 & 5\,38\,24 & \underline{200 + 30 + 2} = 232 \\
a^2 = (200)^2 & \underline{4\,00\,00} & \\
2a + b = 400 + 30 = 430 & 1\,38\,24 & \\
(2a + b)\,b = & \underline{1\,29\,00} & \\
2a + b = 460 + 2 = 462 & 9\,24 & \\
(2a + b)\,b = & \underline{9\,24} &
\end{array}$$

Dividing the given number into groups of two figures each, we learn that its square root will contain hundreds, tens, and units.

Here a, the number of hundreds in the root, is 2; for 4 is the greatest perfect square in 5. Subtract a^2, or 40000. Now $2a$, the trial divisor, is 400; hence b, the number of tens in the root, is $13824 \div 400$, or 3.

$$\therefore\ 2a + b = 400 + 30 = 430, \text{ the true divisor.}$$

Multiply this by b, or 30, and subtract the product. Then in all we have subtracted $(a + b)^2$; that is, $(230)^2$, since $a = 200$, and $b = 30$.

Now let $a = 230$, the *part of the root already found*, of which the square has been subtracted. Then $2a$, the trial divisor, is 460; and b, the units' figure of the root, is $924 \div 460$, or 2.

$$\therefore\ 2a + b = 460 + 2 = 462, \text{ the true divisor.}$$

Multiply this by b, or 2, and subtract the product. Then in all we have subtracted $(a + b)^2$; that is, $(232)^2$, since $a = 230$, and $b = 2$.

Hence, as there is no remainder, the given number is equal to $(232)^2$, and its square root is 232.

Omitting the ciphers, and in each remainder writing the next group of figures only, the work will stand as below:

$$\begin{array}{r|l}
\multicolumn{1}{r}{} & 5\ 38\ 24\,|\underline{232} \\
\multicolumn{1}{r}{} & 4 \\
43 & 1\ 38 \\
 & \underline{1\ 29} \\
462 & 9\ 24 \\
 & \underline{9\ 24}
\end{array}$$

Example 2. Find the square root of 1522756.

Formula, $(a + b)^2 = a^2 + (2a + b)b.$

$$\begin{array}{r|l}
\multicolumn{1}{r}{} & 1\ 52\ 27\ 56\,|\underline{1234} \\
\multicolumn{1}{r}{} & \underline{1} \\
22 & 52 \\
 & \underline{44} \\
243 & 8\ 27 \\
 & \underline{7\ 29} \\
2464 & 98\ 56 \\
 & \underline{98\ 56}
\end{array}$$

133. If a number has decimal places, its *square* will have *twice* as many. Thus $0.8^2 = 0.64$; $0.25^2 = 0.0625$.

Hence to determine how many decimal figures there will be in the square root of a number, we divide its decimal figures into groups of two figures each, beginning at the decimal point. If the group to the right does not contain two figures, a cipher must be annexed.

EXAMPLE. Find the square root of 7527.5.

Formula, $(a + b)^2 = a^2 + (2a + b)b.$

$$\begin{array}{r|l}
 & 75\ 27.50\ \underline{|86.76+} \\
 & \underline{64} \\
166 & 11\ 27 \\
 & \underline{9\ 96} \\
1727 & 1\ 31\ 50 \\
 & \underline{1\ 20\ 89} \\
17346 & 10\ 61\ 00 \\
 & \underline{10\ 40\ 76} \\
 & \qquad 20\ 24
\end{array}$$

Here at first $a = 80$, and $b = 6$; next $a = 86$, and $b = 0.7$; next $a = 86.7$, and $b = 0.06$. By adding other groups of ciphers we can approximate the root as nearly as we please.

Exercise 29.

Find the square root of

1. 2916.
2. 2601.
3. 17956.
4. 33489.
5. 119025.
6. 15129.
7. 103041.
8. 835396.
9. 29376400.
10. 52.2729.
11. 53.7289.
12. 883.2784.
13. 1.97262025.
14. 3080.25.
15. 41.2164.
16. 384524.01.
17. 0.0022448644.
18. 0.68112009.

Find to three places of decimals the square root of

19. 5.
20. 50.
21. 0.5.
22. 0.3.
23. 0.03.
24. 0.004.
25. 2.5.
26. 0.005.
27. 111.
28. $\frac{2}{3}$.

$$\sqrt{\tfrac{2}{3}} = \sqrt{0.6666+} = 0.8164+.$$

Or $$\sqrt{\tfrac{2}{3}} = \sqrt{\tfrac{6}{9}} = \tfrac{1}{3}\sqrt{6}$$
$$= \tfrac{1}{3}(2.4494+) = 0.8164+.$$

29. $\frac{5}{6}$. **30.** $\frac{1}{3}$. **31.** $\frac{1}{2}$. **32.** $\frac{4}{5}$. **33.** $\frac{8}{11}$.

CUBE ROOT OF POLYNOMIALS.

134. By § 113, we know that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Hence the cube of a binomial has *four* terms; and, when these are arranged according to the powers of some letter, the extreme terms are the cubes of the terms of the binomial.

Conversely, if a perfect cube consists of four terms arranged according to the powers of some letter, its cube root is the sum of the cube roots of its extreme terms.

Thus, if $64a^3 - 144a^2b + 108ab^2 - 27b^3$ is a perfect cube, its cube root is $4a - 3b$; for the expression has four terms which are arranged according to the powers of a, and the cube roots of its extreme terms are $4a$ and $-3b$ respectively.

Again, if $27a^6 - 54a^5b + 36a^4b^2 - 8a^3b^3$ is a perfect cube, its cube root is $3a^2 - 2ab$. The expression is a perfect cube; for,

$$(3a^2 - 2ab)^3 = 27a^6 - 54a^5b + 36a^4b^2 - 8a^3b^3.$$

Any perfect cube that contains only *three* different powers of a particular letter can be arranged according to the powers of that letter, and its cube root obtained by inspection, as above.

For example, if we arrange the expression,

$$a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 6abc + 3b^2c + 3bc^2,$$

according to its three different powers of a, we have

$$a^3 + 3a^2(b+c) + 3a(b^2 + c^2 + 2ab) + (b^3 + 3b^2c + 3bc^2 + c^3),$$

or $$a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3,$$

of which the cube root is seen to be $a + b + c$.

Exercise 30.

Find the cube root of

1. a^3+3a^2+3a+1.
3. $8m^3-12m^2+6m-1$.
2. $x^3+6x^2+12x+8$.
4. $a^3x^3-3a^2x^2y^2+3axy^4-y^6$.
5. $64a^3+108ab^2-144a^2b-27b^3$.
6. $x^3-24x^2y+192xy^2-512y^3$.
7. $a^3+6a^2b-3a^2c+12ab^2-12abc+3ac^2+8b^3-12b^2c+6bc^2-c^3$.
8. $\frac{x^3}{8}-\frac{3x^2}{4}+\frac{3x}{2}-1$.
10. $8x^3-4x^2y^2+\frac{2}{3}xy^4-\frac{y^6}{27}$.
9. $\frac{x^3}{27}+\frac{2x^2}{3}+4x+8$.
11. $\frac{27x^3}{64y^3}-\frac{27x^2}{8y^2}+\frac{9x}{y}-8$.

135. To find the cube root of any polynomial.

The rule is given by the identity,

$$(a+b)^3 \equiv a^3+(3a^2+3ab+b^2)b.$$

At first a represents the first term of the root, and b the second term. After the first two terms of the root have been obtained, a denotes the part of the root already found, and b the next term of the root. $3a^2$ denotes the *trial* divisor for obtaining b; and $3a^2+3ab+b^2$, the *true* divisor.

The work may be arranged as below:

$$\begin{array}{rl|l}
 & a^3+3a^2b+3ab^2+b^3 & \underline{a+b} \\
 & \underline{a^3\qquad\qquad\qquad\qquad} & \\
3a^2 & \\
3ab+b^2 & \left|\begin{array}{l}3a^2b+3ab^2+b^3 \\ \underline{3a^2b+3ab^2+b^3}\end{array}\right. &
\end{array}$$

Now a, the first term of the root, is the cube root of a^3, the first term of the given expression.

If a^3 is subtracted, the remainder is $3a^2b + 3ab^2 + b^3$. Hence b, the next term of the root, is obtained by dividing the first term of this remainder by $3a^2$, that is, by *three times the square of the part of the root already found.*

Again, if to $3a^2$, the *trial* divisor, we add $3ab + b^2$, we obtain the *true* divisor $3a^2 + 3ab + b^2$.

Example. Find the cube root of

$$8x^6 - 36x^5 - 63x^3 + 66x^4 - 9x + 33x^2 + 1.$$

Arrange the terms according to the descending powers of x, and for convenience place the root above the expression.

$$\begin{array}{rr|l}
 & & \underline{2x^2-3x+1} \\
 & & 8x^6-36x^5+66x^4-63x^3+33x^2-9x+1 \\
 & & \underline{8x^6} \\
3a^2= & 12x^4 & -36x^5+66x^4-63x^3 \\
3ab+b^2= & \underline{-18x^3+9x^2} & \\
 & 12x^4-18x^3+9x^2 & \underline{-36x^5+54x^4-27x^3} \\
3a^2= & 12x^4-36x^3+27x^2 & 12x^4-36x^3+33x^2-9x+1 \\
3ab+b^2= & \underline{6x^2-9x+1} & \\
 & 12x^4-36x^3+33x^2-9x+1 & \underline{12x^4-36x^3+33x^2-9x+1}
\end{array}$$

Here a, the first term of the root, is $\sqrt[3]{8x^6}$, or $2x^2$.

Subtract a^3, or $8x^6$. Now $3a^2$, the *trial* divisor, is $12x^4$; hence b, the second term of the root, is $(-36x^5) \div 12x^4$, or $-3x$.

$$\therefore\ 3ab + b^2 = 3(2x^2)(-3x) + (-3x)^2 = -18x^3 + 9x^2.$$

$$\therefore\ 3a^2 + 3ab + b^2 = 12x^4 - 18x^3 + 9x^2, \text{ the \textit{true} divisor.}$$

Multiply this by b, or $-3x$, and subtract the product. Then in all we have subtracted $a^3 + (3a^2 + 3ab + b^2)b$, or $(a+b)^3$; that is, we have subtracted $(2x^2 - 3x)^3$, since $a = 2x^2$, and $b = -3x$.

Next, let $a = 2x^2 - 3x$, *the part of the root already found*, of which the cube has been subtracted.

Then $$3a^2 = 3(2x^2 - 3x)^2$$
$$= 12x^4 - 36x^3 + 27x^2, \text{ the \textit{trial} divisor.}$$

Hence b, the next term of the root, is $(12x^4) \div (12x^4)$, or 1.

$$\therefore\ 3ab + b^2 = 3(2x^2 - 3x)\cdot 1 + 1^2 = 6x^2 - 9x + 1.$$

$$\therefore\ 3a^2 + 3ab + b^2 = 12x^4 - 36x^3 + 33x^2 - 9x + 1, \text{ the \textit{true} divisor.}$$

Multiply this by b, or 1, and subtract the product. Then in all we have subtracted $(a+b)^3$, or $(2x^2-3x+1)^3$, since $a=2x^2-3x$ and $b=1$. As there is no remainder, the required cube root is $2x^2-3x+1$.

Exercise 31.

Find the cube root of

1. $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$.
2. $y^6-3y^5+6y^4-7y^3+6y^2-3y+1$.
3. $1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$.
4. $8a^6-36a^5+66a^4-63a^3+33a^2-9a+1$.
5. $8x^6+12x^5-30x^4-35x^3+45x^2+27x-27$.
6. $27x^6-27x^5-18x^4+17x^3+6x^2-3x-1$.
7. $24x^4y^2+96x^2y^4-6x^5y+x^6-96xy^5+64y^6-56x^3y^3$.
8. $27x^6-54x^5a+117x^4a^2-116x^3a^3+117x^2a^4-54xa^5+27a^6$.
9. $216+342x^2+171x^4+27x^6-27x^5-109x^3-108x$.
10. $x^3-9x+\frac{27}{x}-\frac{27}{x^3}$.
11. $\frac{x^6}{y^3}-6x^4+12x^2y^3-8y^6$.
12. $\frac{x^3}{y^3}+\frac{6x^2}{y^2}+\frac{9x}{y}-4-\frac{9y}{x}+\frac{6y^2}{x^2}-\frac{y^3}{x^3}$.
13. $\frac{x^3}{27}-\frac{x^2}{3}+2x-7+\frac{18}{x}-\frac{27}{x^2}+\frac{27}{x^3}$.
14. $\frac{6b}{a}+\frac{6a}{b}-7+\frac{a^3}{b^3}-\frac{3a^2}{b^2}-\frac{3b^2}{a^2}+\frac{b^3}{a^3}$.
15. $\frac{60x^4}{y^4}-\frac{80x^3}{y^3}-\frac{90x^2}{y^2}+\frac{8x^6}{y^6}+\frac{108x}{y}-27+\frac{48x^5}{y^5}$.

136. Cube Root of Arithmetical Numbers. Since $\sqrt[3]{1} = 1$, and $\sqrt[3]{1000} = 10$, it follows that the cube root of any number between 1 and 1000 lies between 1 and 10; that is, if a number contains one, two, or three integral figures, its cube root will contain one integral figure. Again, $\sqrt[3]{1000} = 10$, and $\sqrt[3]{1000000} = 100$; hence, if a number contains four, five, or six integral figures, its cube root will contain two integral figures; and so on.

Hence, to determine how many integral figures there will be in the cube root of a number, we divide its integral figures into groups of three figures each, beginning at units' place. The last group to the left may contain only one or two figures.

When the figures of a number have been divided into groups of three figures each, the rule for finding its cube root is given by the formula,

$$(a+b)^3 \equiv a^3 + (3a^2 + 3ab + b^2)b,$$

in which a denotes tens with reference to b.

Example. Find the cube root of 614125.

$$\begin{array}{rrr|ll}
\multicolumn{3}{r}{} & \multicolumn{1}{l}{614\,125} & \multicolumn{1}{|l}{\underline{80+5=85}} \\
\multicolumn{1}{r}{a^3 =} & \multicolumn{2}{r}{} & \multicolumn{1}{l}{\underline{512\,000}} & \\
3a^2 = & 3(80)^2 = & 19200 & 102\,125 & \\
3ab = & 3\cdot 80\cdot 5 = & 1200 & & \\
b^2 = & 5^2 = & \underline{25} & & \\
3a^2 + 3ab + b^2 = & & 20425 & \underline{102\,125} &
\end{array}$$

Here a, the number of tens in the root, is 8; for 512, or 8^3, is the greatest perfect cube in 614. Subtract a^3, or 80^3. Now $3a^2$, the *trial* divisor, is $3(80)^2$, or 19200; hence b, the number of units in the root, is $102125 \div 19200$, or 5.

Hence $\quad 3a^2 + 3ab + b^2 = 20425$, the *true* divisor.

Multiply this by b, or 5; and subtract the result. Then in all we have subtracted $a^3 + (3a^2 + 3ab + b^2)b$, or $(a+b)^3$; that is, we have subtracted 85^3, since $a = 80$ and $b = 5$. As there is no remainder, the given number equals 85^3, and its cube root is 85.

Omitting the ciphers and explanations, the work will stand as below:

$$\begin{array}{rl}
 & 614\ 125\,\lfloor\underline{85} \\
 & 512 \\ \cline{2-2}
19200 & |\,102\ 125 \\
1200 & | \\
25 & | \\ \cline{1-1}
20425 & |\,102\ 125 \\ \cline{2-2}
\end{array}$$

137. If a number has decimal places, its *cube* will have *three* times as many. Thus $0.2^3 = 0.008$; $0.12^3 = 0.001728$. Hence, to determine how many decimal figures there will be in the *cube root* of a number, we divide its decimal figures into groups of three figures each, beginning at the decimal point. If the group to the right does not contain three figures, ciphers must be annexed.

Example. Find the cube root of 129554.6.

Formula, $(a + b)^3 = a^3 + (3a^2 + 3ab + b^2)b$.

$$\begin{array}{rl}
 & 129\ 554.600\,\lfloor\underline{50.6} \\
 & 125 \\ \cline{2-2}
750000 & |\,4\ 554\ 600 \\
9000 & | \\
36 & | \\ \cline{1-1}
759036 & |\,4\ 554\ 216 \\ \cline{2-2}
 & 384
\end{array}$$

Here at first $a = 50$, and $b = 0$; next, $a = 50.0$, and $b = 0.6$. In this example the exact root cannot be found; but by adding groups of ciphers, we can approximate it as nearly as we please.

Exercise 32.

Find the cube root of

1. 74088.
2. 15625.
3. 32768.
4. 110592.
5. 262144.
6. 1481544.
7. 103.823.
8. 884.736.
9. 1953125.

10. 7077888. **12.** 12.812904. **14.** 264.609288.

11. 2.803221. **13.** 56.623104. **15.** 1076890625.

Extract to three figures the cube root of

16. 2.5. **18.** 0.01. **20.** 0.08. **22.** $\frac{5}{7}$.

17. 0.2. **19.** 4. **21.** $\frac{1}{8}$. **23.** $\frac{6}{11}$.

138. Higher Roots. The rules for finding the *fourth, fifth, sixth,* etc., roots of an expression or number are given by the identities,

$$(a+b)^4 \equiv a^4 + (4a^3 + 6a^2b + 4ab^2 + b^3)b, \qquad (1)$$

$$(a+b)^5 \equiv a^5 + (5a^4 + 10a^3b + 10a^2b^2 + 5ab^3 + b^4)b, \qquad (2)$$

. . . .

The fourth root of an expression can also be obtained by finding the square root of its square root; and the sixth root of an expression can be obtained by finding the cube root of its square root; and so on.

From formula (1), we see that if a perfect fourth power contains only five terms arranged according to the powers of some letter, its fourth root is the sum or the difference of the fourth roots of its extreme terms.

Thus, if $81x^4 + 108x^3 + 54x^2 + 12x + 1$ is a perfect fourth power, its fourth root is $3x + 1$. It is a perfect fourth power, for

$$(3x + 1)^4 = 81x^4 + 108x^3 + 54x^2 + 12x + 1.$$

From formula (2), we see that if a perfect fifth power contains only six terms arranged according to the powers of some letter, its fifth root is the sum of the fifth roots of its extreme terms; and so on for the other powers.

Thus, if $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$ is a perfect fifth power, its fifth root is $2x - 1$. It is a perfect fifth power, for

$$(2x - 1)^5 = 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1.$$

Again, if $729 - 1458x + 1215x^2 - 540x^3 + 135x^4 - 18x^5 + x^6$ is a perfect sixth power, its sixth root is $3 - x$.

Exercise 33.

Find the fourth root of

1. $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.
2. $x^4 - 8x^3a + 24x^2a^2 - 32xa^3 + 16a^4$.
3. $1 + 4a + 4a^7 + 10a^6 + a^8 + 10a^2 + 16a^3 + 16a^5 + 19a^4$.

Find the fifth root of

4. $80a^2x^3 - 80ax^4 + 32x^5 - 40a^3x - a^5 + 10a^4x$.
5. $90ax^3 - 15ax^4 + x^5 - 270a^3x^2 + 405a^4x - 243a^5$.

Find the sixth root of

6. $192x + 64 + 240x^2 + x^6 + 12x^5 + 60x^4 + 160x^3$.
7. $1215a^4 - 1458a^5 - 540a^3 + 135a^2 - 18a + 1 + 729a^6$.
8. $60a^2x^4 - 160a^3x^3 + 64a^6 + x^6 - 12ax^5 + 240a^4x^2 - 192a^5x$.

139. Quadratic Imaginary Numbers. We shall assume that the commutative law of multiplication holds for imaginary numbers.

By definition we have

$$-1 = \sqrt{-1} \times \sqrt{-1}, \tag{1}$$

and

$$a = \sqrt{a} \times \sqrt{a}, \tag{2}$$

in which a is positive.

Multiplying (1) by (2), member by member, we have

$$-a = \sqrt{a}\sqrt{-1} \times \sqrt{a}\sqrt{-1}.$$

$$\therefore \sqrt{-a} = \sqrt{a}\sqrt{-1}. \tag{3}$$

By law (3) any quadratic imaginary term may be reduced to the form $c\sqrt{-1}$, in which c denotes some real number. The factor $\sqrt{-1}$ is called the **Imaginary Factor** or **Unit.**

Thus, by law (3) we have

(1) $\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3} \times \sqrt{-1}.$

(2) $\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9} \times \sqrt{-1} = 3\sqrt{-1}.$

(3) $\sqrt{-4x^2} = \sqrt{4x^2(-1)} = \sqrt{4x^2} \times \sqrt{-1} = 2x\sqrt{-1}.$

(4) $\sqrt{-\frac{1}{9}} = \sqrt{\frac{1}{9}(-1)} = \sqrt{\frac{1}{9}} \times \sqrt{-1} = \frac{1}{3}\sqrt{-1}.$

Exercise 34.

Reduce the following expressions to the form $c\sqrt{-1}$:

1. $\sqrt{-4}.$ **6.** $\sqrt{-81}.$ **11.** $\sqrt{-9a^2}.$

2. $\sqrt{-25}.$ **7.** $\sqrt{-169}.$ **12.** $\sqrt{-16x^4}.$

3. $\sqrt{-36}.$ **8.** $\sqrt{-289}.$ **13.** $\sqrt{-(x^2+y^2)^2}.$

4. $\sqrt{-49}.$ **9.** $\sqrt{-625}.$ **14.** $\sqrt{-(x+a)}.$

5. $\sqrt{-100}.$ **10.** $\sqrt{-\frac{1}{16}}.$ **15.** $\sqrt{-(x-a)^4}.$

16. Add $\sqrt{-25}$, $\sqrt{-16}$, and $-\sqrt{-64}.$

$$\begin{aligned}\sqrt{-25}+\sqrt{-16}-\sqrt{-64} &= 5\sqrt{-1}+4\sqrt{-1}-8\sqrt{-1}\\ &= (5+4-8)\sqrt{-1} = \sqrt{-1}.\end{aligned}$$

17. Add $\sqrt{-36}$, $-\sqrt{-49}$, $-\sqrt{-100}$, and $\sqrt{-4}.$

18. Add $\sqrt{-9}$, $\sqrt{-16}$, $-\sqrt{-25}$, and $-\sqrt{-49}.$

19. Add $\sqrt{-81}$, $-\sqrt{-64}$, $-\sqrt{-121}$, and $-\sqrt{-144}.$

20. Add $\sqrt{-9a^2}$, $-\sqrt{-4a^2}$, $-\sqrt{-16a^2}$, and $\sqrt{-36a^2}.$

21. Add $a-b\sqrt{-1}$ and $a+b\sqrt{-1}.$

22. Add $2+\sqrt{-3}$ and $2-\sqrt{-3}.$

23. Multiply $\sqrt{-5}$ by $\sqrt{-7}$.

$$\begin{aligned} \sqrt{-5} &= \sqrt{5} \times \sqrt{-1} \\ \sqrt{-7} &= \sqrt{7} \times \sqrt{-1} \\ \hline \text{Product} &= \sqrt{35} \times (-1) = -\sqrt{35}. \end{aligned}$$

For $\sqrt{-1} \times \sqrt{-1} = -1$ by (1) of § 139.

Multiply

24. $\sqrt{-4}$ by $\sqrt{-9}$.

25. $\sqrt{-16}$ by $\sqrt{-25}$.

26. $\sqrt{-7}$ by $\sqrt{-15}$.

27. $\sqrt{-13}$ by $\sqrt{-20}$.

28. $\sqrt{-a^2}$ by $\sqrt{-b^2}$.

29. $\sqrt{-9x^2}$ by $\sqrt{-16y^2}$.

30. $\sqrt{-7a}$ by $\sqrt{-5b}$.

31. $\sqrt{-4a^2x}$ by $\sqrt{-9b^2y}$.

32. $-3+\sqrt{-2}$ by $-3+\sqrt{-2}$.

$$\begin{aligned} (-3+\sqrt{-2})(-3-\sqrt{-2}) &= (-3)^2 - (\sqrt{-2})^2 \\ &= 9-(-2) = 11. \end{aligned}$$

33. $2+\sqrt{-3}$ by $2-\sqrt{-3}$.

34. $5-\sqrt{-7}$ by $5+\sqrt{-7}$.

35. $-1-\sqrt{-5}$ by $-1+\sqrt{-5}$.

36. $-7-\sqrt{-11}$ by $-7+\sqrt{-11}$.

CHAPTER X.

IMPORTANT THEOREMS IN MULTIPLICATION AND DIVISION.

140. *The product of the sum and the difference of any two terms is equal to the difference of their squares.*

For let A and B represent any two terms; then $A+B$ will denote their sum, and $A-B$ their difference. Now, by the distributive law, we have

$$\begin{aligned}(A+B)(A-B)&=(A+B)A+(A+B)(-B)\\&=A^2-B^2.\end{aligned}$$

EXAMPLE 1. Find the product of $2x^2+5by^3$ and $2x^2-5by^3$.

$$\begin{aligned}(2x^2+5by^3)(2x^2-5by^3)&=(2x^2)^2-(5by^3)^2 \qquad \S\ 140\\&=4x^4-25b^2y^6.\end{aligned}$$

EXAMPLE 2. Find the product of $a+b+c$ and $a+b-c$.

$$\begin{aligned}(a+b+c)(a+b-c)&=[(a+b)+c][(a+b)-c]\\&=(a+b)^2-c^2 \qquad \S\ 140\\&=a^2+2ab+b^2-c^2.\end{aligned}$$

Here we first write $a+b+c$ and $a+b-c$ as the sum and the difference of the two terms $(a+b)$ and c.

EXAMPLE 3. Find the product of $a+b-c$ and $a-b+c$.

$$\begin{aligned}(a+b-c)(a-b+c)&=[a+(b-c)][a-(b-c)]\\&=a^2-(b-c)^2\\&=a^2-b^2+2bc-c^2.\end{aligned}$$

Exercise 35.

By § 140 write out each of the following products:

1. $(a+b)(a-b)$.
2. $(x+5)(x-5)$.
3. $(1+3x)(1-3x)$.
4. $(a^2+b^2)(a^2-b^2)$.
5. $(x^2+4y^3)(x^2-4y^3)$.
6. $(3x^2+5y^2)(3x^2-5y^2)$.
7. $(2ax+3by)(2ax-3by)$.
8. $(4cx^2+5b^2y)(4cx^2-5b^2y)$.
9. $(a+b+c)(a-b-c)$.
10. $(1+b-c)(1-b+c)$.
11. $(a-b+c)(a-b-c)$.
12. $(x+3y-2z)(x-3y+2z)$.
13. $(x^2+xy+y^2)(x^2-xy+y^2)$.
14. $(y^2+y+2)(y^2-y+2)$.
15. $(3a+b-3c)(3a-b+3c)$.
16. $(2a-\sqrt{3})(2a+\sqrt{3})$.
17. $(2x-\frac{1}{2}\sqrt{a-c})(2x+\frac{1}{2}\sqrt{a-c})$.
18. $(-3a+7x)(-3a-7x)$, or
    $[(-3a)+7x][(-3a)-7x]$.
19. $(-\frac{1}{2}p+\frac{1}{2}\sqrt{p^2-4q})(-\frac{1}{2}p-\frac{1}{2}\sqrt{p^2-4q})$.

141. *The product of two binomials, such as* x + a *and* x + b, *is equal to the square of the first term, plus the sum of the second terms into the first term, plus the product of the second terms.*

For whatever be the values of x, a and b, we have

$$\begin{aligned}(x+a)(x+b) &= (x+a)\,x+(x+a)\,b\\ &= x^2+(a+b)\,x+ab.\end{aligned}$$

Thus

$$\begin{aligned}(x+7)(x+5)&=x^2+(7+5)x+7\times 5\\&=x^2+12x+35.\\(x-7)(x-5)&=[x+(-7)][x+(-5)]\\&=x^2+(-7-5)x+(-7)(-5)\\&=x^2-12x+35.\\(x+7)(x-5)&=x^2+(7-5)x+7(-5)\\&=x^2+2x-35.\\(x-7)(x+5)&=x^2-2x-35.\end{aligned}$$

Exercise 36.

Write out the product of

1. $(x+8)(x+5)$.

2. $(x-3)(x+10)$.

3. $(x+7)(x-9)$.

4. $(x-4)(x+11)$.

5. $(a+9)(a-5)$.

6. $(a-8)(a+4)$.

7. $(a-6)(a+13)$.

8. $(x-3a)(x+2a)$.

9. $(a-9b)(a-8b)$.

10. $(3x-2y)(3x+y)$.

$$\begin{aligned}(3x-2y)(3x+y)&=(3x)^2+(-2y+y)3x+(-2y)y\\&=9x^2-3xy-2y^2.\end{aligned}$$

11. $(a-5b)(a+10b)$.

12. $(x^2-6)(x^2+4)$.

13. $(a^2+2x)(a^2-5x)$.

14. $(xy-9)(xy+6)$.

15. $(xy-6ab)(xy+2ab)$.

16. $(ab-5)(ab+7)$.

17. $(xy-7ab)(xy-2ab)$.

18. $(x-4ab)(x+5ab)$.

19. $(xz-9ab)(xz+11ab)$.

20. $[x^2+4(y+z)][x^2-5(y+z)]$.

$$[x^2+4(y+z)][x^2-5(y+z)]=x^4-(y+z)x^2-20(y+z)^2.$$

21. $(x+y+3)(x+y-5)$.

22. $(a+b-7)(a+b-8)$.

23. $(x-y-9)(x-y+8)$.

24. $(a-2b-4)(a-2b+6)$.

142. *The difference of the same powers of two numbers is divisible by the difference of the numbers.*

For by division we obtain

$$\frac{x^4-a^4}{x-a} \equiv x^3+x^2a+xa^2+a^3.$$

By inspection we find in this quotient the following laws of exponents and coefficients:

The exponent of x *in the first term is* 1 *less than that of* x *in the dividend, and it decreases by* 1 *in each succeeding term.*

The exponent of a *is* 1 *in the second term and increases by* 1 *in each succeeding term.*

The number of terms is equal to the exponent of x *in the dividend, and the coefficient of each term is* $+1$.

Let n denote any positive integer; then, if these laws hold true in general, we shall have

$$\frac{x^n-a^n}{x-a} \equiv x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1}. \quad (1)$$

Now multiplying each member of (1) by $x-a$, we obtain the identity

$$x^n-a^n \equiv x^n-a^n.$$

Hence, by § 48, (1) is an identity; therefore x^n-a^n is exactly divisible by $x-a$.

The quotient may be written out by the laws given above or from formula (1).

Thus $$\frac{x^3-a^3}{x-a}=x^2+xa+a^2.$$

EXAMPLE 1. Write out the quotient of $\frac{27a^3-b^3}{3a-b}$.

$$\begin{aligned}\frac{27a^3-b^3}{3a-b} &= \frac{(3a)^3-b^3}{(3a)-b} \\ &= (3a)^2+(3a)b+b^2 \\ &= 9a^2+3ab+b^2.\end{aligned}$$

EXAMPLE 2. Write out the quotient of $\frac{32a^5 - 243x^5}{2a - 3x}$

$$\begin{aligned}\frac{32a^5 - 243x^5}{2a - 3x} &= \frac{(2a)^5 - (3x)^5}{2a - 3x} \\ &= (2a)^4 + (2a)^3(3x) + (2a)^2(3x)^2 + 2a(3x)^3 + (3x)^4 \\ &= 16a^4 + 24a^3x + 36a^2x^2 + 54ax^3 + 81x^4.\end{aligned}$$

143. *The sum* $x^n + a^n$ *is not divisible by* $x - a$.

For evidently $x^n + a^n \equiv (x^n - a^n) + 2a^n$.

Now $x^n - a^n$ is divisible by $x - a$, but $2a^n$ is not; hence $(x^n - a^n) + 2a^n$, or its equal $x^n + a^n$, is not divisible by $x - a$.

Exercise 37.

Write out the quotient of

1. $\frac{x^5 - a^5}{x - a}$.

2. $\frac{x^7 - a^7}{x - a}$.

3. $\frac{x^3y^3 - a^3}{xy - a}$.

4. $\frac{64a^3 - 8x^3}{4a - 2x}$.

5. $\frac{a^6x^6 - 343b^3}{a^2x^2 - 7b}$.

6. $\frac{1 - 729y^3}{1 - 9y}$.

7. $\frac{64a^9b^3 - 8x^6y^3}{4a^3b - 2x^2y}$.

8. $\frac{a^5 - 32}{a - 2}$.

9. $\frac{32x^5 - 1}{2x - 1}$.

10. $\frac{1024a^5 - 32b^5}{4a - 2b}$.

11. $\frac{x^n - 1}{x - 1}$.

12. $\frac{a^3 - (x + y)^3}{a - (x + y)}$.

13. $\frac{(x + 2y)^3 - (y + 2x)^3}{x + 2y - (y + 2x)}$.

14. $\frac{x^5 - (a - b)^5}{x - a + b}$.

144. $x^n - a^n$ *is divisible by* $x + a$ *when* n *is even; and* $x^n + a^n$ *is divisible by* $x + a$ *when* n *is odd.*

For putting $-a$ for a in (1) of § 142, we obtain

$$\frac{x^n - (-a)^n}{x - (-a)} \equiv x^{n-1} + x^{n-2}(-a) + x^{n-3}(-a)^2 + \cdots x(-a)^{n-2} + (-a)^{n-1}. \quad (1)$$

If n is even, $n - 2$ is even, $n - 1$ odd, and (1) becomes

$$\frac{x^n - a^n}{x + a} \equiv x^{n-1} - x^{n-2}a + x^{n-1}a^2 - \cdots + xa^{n-2} - a^{n-1}, \quad (2)$$

which proves the first part of the theorem.

If n is odd, $n - 2$ is odd, $n - 1$ even, and (1) becomes

$$\frac{x^n + a^n}{x + a} \equiv x^{n-1} - x^{n-2}a + x^{n-3}a^2 - \cdots - xa^{n-2} + a^{n-1}, \quad (3)$$

which proves the second part of the theorem.

Note that in each quotient there are n terms, and that the coefficient of each even term is -1.

The laws of exponents are the same as in (1) of § 142.

Thus $\frac{x^3 + a^3}{x + a} = x^2 - xa + a^2.$

Example 1. Write out the quotient of $\frac{8\,a^3b^6 + 216\,c^3x^9}{2\,ab^2 + 6\,cx^3}$.

$$\begin{aligned} \frac{8\,a^3b^6 + 216\,c^3x^9}{2\,ab^2 + 6\,cx^3} &= \frac{(2\,ab^2)^3 + (6\,cx^3)^3}{2\,ab^2 + 6\,cx^3} \\ &= (2\,ab^2)^2 - (2\,ab^2)(6\,cx^3) + (6\,cx^3)^2 \\ &= 4\,a^2b^4 - 12\,ab^2cx^3 + 36\,c^2x^3. \end{aligned}$$

Exercise 38.

Write out the quotient of

1. $\frac{a^3 + b^3}{a + b}$.
2. $\frac{a^5 + b^5}{a + b}$.
3. $\frac{a^7 + b^7}{a + b}$.

4. $\frac{x^3y^3 + 216z^3}{xy + 6z}$.

5. $\frac{a^6b^6 + 8c^3}{a^2b^2 + 2c}$.

6. $\frac{1 + 8a^3}{1 + 2a}$.

7. $\frac{1 + x^5y^5}{1 + xy}$.

8. $\frac{a^5 + 32}{a + 2}$.

9. $\frac{a^5b^5 + 243}{ab + 3}$.

10. $\frac{x^7y^7 + 128}{xy + 2}$.

11. $\frac{729 + 8b^6}{9 + 2b^2}$.

12. $\frac{a^{10}b^{10} + 32x^5}{a^2b^2 + 2x}$.

13. Prove that (2) and (3) in § 144 are identities, by multiplying both members by $x + a$.

14. By the method used in § 143 prove that $x^n - a^n$ is not divisible by $x + a$ when n is *odd*, and that $x^n + a^n$ is not divisible by $x + a$ when n is even.

Write out the quotient of

15. $\frac{a^4b^8 - x^8y^{12}}{ab^2 + x^2y^3}$.

16. $\frac{a^6b^{12} - x^{12}y^{18}}{ab^2 + x^2y^3}$.

17. $\frac{16x^4y^4 - 256a^8b^8}{2xy + 4a^2b^2}$.

18. $\frac{64a^6b^{12} - 729x^6y^{18}}{2ab^2 + 3xy^3}$.

CHAPTER XI.

FACTORING.

145. By the *Factors* of a given expression, we shall in this chapter usually mean its rational factors.

146. Monomial Factors. Any monomial is readily resolved into its monomial factors.

Thus the factors of $35x^2b$ are 5, 7, x, x, and b.

A factor common to all the terms of any expression is, by § 54, a factor of the expression.

EXAMPLE. Factor $3ax^2 + 6a^2x - 9a^3x^3$.

By inspection $3ax$ is seen to be a factor of each term; hence by § 54 we have

$$\begin{aligned}3ax^2 + 6a^2x - 9a^3x^3 &\equiv (3ax)x + (3ax)2a + (3ax)(-3a^2x^2)\\ &\equiv 3ax(x + 2a - 3a^2x^2).\end{aligned}$$

That is, the required factors are 3, a, x, and $x + 2a - 3a^2x^2$.

Exercise 39.

Factor

1. $15ab^2 - 45a^2b - 30a^2b^2x^2$.

   Ans. 3, 5, a, b, and $b - 3a - 2abx^2$.

2. $x^2 + 5x$.
3. $4a^2 - 6a^2b$.
4. $2ax + \frac{1}{2}x^2$.
5. $7a^3 - 21a^2b$.
6. $x^4 - 5x^3y + 20x^2y^2$.
7. $5ax^2 - 10a^2x - 5a^2x^2$.
8. $38a^3b^5 - 57a^4b^2$.
9. $3a^3b - 6a^2b^2 + 9a^2b^3$.
10. $15a^3b - 10a^3c + 5a^3d$.
11. $8a^2x^2 - 4a^2y - 12a^2b$.

12. $2a^ny^n + 6a^{n+2}y^{n+1}$. *Ans.* 2, a^n, y^n, and $1 + 3a^2y$.

13. $ax^{m+2}y^{n+1} + bx^{m+3}y^{n+2} + cx^my^n$.

Ans. x^m, y^n, and $ax^2y + bx^3y^2 + c$.

147. Polynomials that are *Perfect Powers* may be factored by the principles of Chapter IX.

Any expression that is to be factored should be first examined for monomial factors.

EXAMPLE 1. Factor $-3a^5 + 30a^4b^3 - 75a^3b^6$.

$$-3a^5 + 30a^4b^3 - 75a^3b^6 \equiv -3a^3(a^2 - 10ab^3 + 25b^6)$$
$$\equiv -3a^3(a - 5b^3)^2. \qquad \text{§ 128}$$

EXAMPLE 2. Factor $18a^2 + 36ab + 18b^2 - 12ac - 12bc + 2c^2$.

The expression
$$\equiv 2(9a^2 + 18ab + 9b^2 - 6ac - 6bc + c^2)$$
$$\equiv 2[(3a)^2 + 2(3a)(3b - c) + (3b - c)^2]$$
$$\equiv 2(3a + 3b - c)^2.$$

EXAMPLE 3. Factor $8a^2c^3 - 36a^2c^2x + 54a^2cx^2 - 27a^2x^3$.

The expression
$$\equiv a^2(8c^3 - 36c^2x + 54cx^2 - 27x^3)$$
$$\equiv a^2(2c - 3x)^3.$$

Exercise 40.

Factor

1. $5a^4 - 10a^2b + 5b^2$.
2. $a^3 - 6a^2b + 9ab^2$.
3. $4x^2y^2 - x^4 - 4y^4$.
4. $8x^2 - 4x^4 - 4$.
5. $x^2y^2 + x^3y + \frac{1}{4}xy^3$.
6. $4xy^3 - 4x^2y^2 + x^3y$.
7. $121x^2 + 81y^2 + 198xy$.
8. $3a^3 - 9a^2b + 9ab^2 - 3b^3$.
9. $5a^2x^6 + 15a^2x^4y + 15a^2x^2y^2 + 5a^2y^3$.
10. $24b^2x^3 - 36b^2x^2 + 18b^2x - 3b^2$.
11. $a^2 + 2ab + 4c^2 + 4ac + 4bc + b^2$.
12. $2ax^4 + 4ax^2y^2 - 4ay^2z^2 + 2az^4 + 2ay^4 - 4ax^2z^2$.

13. $3bx^4 - 6bx^2y + 12a^2b - 12aby + 3by^2 + 12abx^2$.

14. $a^2x^3 - 9a^2x^2y + 27a^2xy^2 - 27a^2y^3$.

15. $x^3 + 3x^2y + 3xy^2 - 3ax^2 + y^3 - 3ay^2 - 6axy + 3a^2y + 3a^2x - a^3$.

148. Binomials of the Form $x^2 - a^2$. By § 140 we have

$$x^2 - a^2 \equiv (x + a)(x - a).$$

That is, *the difference between the squares of any two numbers is equal to the product of their sum and difference.*

EXAMPLE 1. Factor $9a^6b^6 - 4c^2$.

$$\begin{aligned} 9a^6b^6 - 4c^2 &\equiv (3a^3b^3)^2 - (2c)^2 \\ &\equiv (3a^3b^3 + 2c)(3a^3b^3 - 2c). \end{aligned}$$

EXAMPLE 2. Factor $9x^2 + 12ab - 9a^2 - 4b^2$.

Writing the expression as the difference of two squares, we have

$$\begin{aligned} 9x^2 + 12ab - 9a^2 - 4b^2 &\equiv 9x^2 - (9a^2 - 12ab + 4b^2) \\ &\equiv (3x)^2 - (3a - 2b)^2 \\ &\equiv [3x - (3a - 2b)][3x + (3a - 2b)] \\ &\equiv (3x - 3a + 2b)(3x + 3a - 2b). \end{aligned}$$

Exercise 41.

Factor

1. $a^2 - 9$.
2. $25a^2 - b^2$.
3. $16 - b^2$.
4. $x^2 - 9y^2$.
5. $64x^2 - 49b^2$.
6. $9a^2 - 16b^2$.
7. $81y^2 - 9x^2$.
8. $36x^2 - 49y^2$.
9. $4a^2b^2 - 9c^2$.
10. $4xy^2 - 9x^3$.
11. $8ab^2 - 18a^3$.
12. $108x^3 - 3x^5$.
13. $7x^5 - 28x^9$.
14. $32xy^3 - 8x^3y$.
15. $7xyz^2 - 7x^3y^3$.
16. $a^2 + 2ab + b^2 - c^2$.

17. $a^2 - 2ab + b^2 - c^2$.

18. $a^2 - b^2 - 2bc - c^2$.

19. $a^2 - b^2 + 2bc - c^2$.

20. $x^2 + 4xy - a^2 + 4y^2$.

21. $x^2 - 1 + 10cx + 25c^2$.

22. $1 + 2ab - a^2 - b^2$.

23. $49x^2 - 1 + 14xy + y^2$.

24. $a^2 - 16x^2 + 6ab + 9b^2$.

25. $x^2 - 9y^2 + 10ax + 25a^2$.

26. $b^2 - a^2 - 4x^2 + 4ax$.

27. $9c^2 - 4x^2 - 9a^2 + 12ax$.

28. $4a^2 - y^2 - 9z^2 + 6yz$.

29. $c^2 - 25a^2 - 9b^2 + 30ab$.

30. $a^2 + b^2 + 2ab - c^2 - d^2 - 2cd$.

31. $a^2 + b^2 - 2ab - x^2 - y^2 - 2xy$.

32. $m^2 + n^2 - 2mn - a^2 - b^2 + 2ab$.

33. $a^2 + n^2 - 2an - b^2 - m^2 - 2bm$.

34. $16a^2 + 8ax + x^2 - 2by - b^2 - y^2$.

35. $9a^2 + 12ab + 4b^2 - (c + x - 2y)^2$.

36. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc - x^2 - y^2 + 2xy$.

Resolve into factors and simplify

37. $(x + 3y)^2 - 4y^2$.

38. $9a^2 - (3a - 5b)^2$.

39. $(5x + 2y)^2 - (3x - y)^2$.

40. $(2x + a - 3)^2 - (3 - 2x)^2$.

41. $(2a + b - c)^2 - (a - b + c)^2$.

149. A Trinomial of the form $x^4 + ax^2y^2 + y^4$ can often be factored by writing it as *the difference of two squares.*

Example 1. Factor $x^4 + x^2a^2 + a^4$.

Adding and subtracting x^2a^2, we obtain

$$\begin{aligned} x^4 + x^2a^2 + a^4 &\equiv x^4 + 2x^2a^2 + a^4 - x^2a^2 \\ &\equiv (x^2 + a^2)^2 - (xa)^2 \\ &\equiv (x^2 + a^2 + xa)(x^2 + a^2 - xa). \end{aligned}$$

Example 2. Factor $4x^4 - 21x^2a^2 + 9a^4$.

Separating the term $-21x^2a^2$ into the two parts $-12x^2a^2$ and $-9x^2a^2$, we have

$$\begin{aligned}4x^4 - 21x^2a^2 + 9a^4 &\equiv 4x^4 - 12x^2a^2 + 9a^4 - 9x^2a^2\\ &\equiv (2x^2 - 3a^2)^2 - (3xa)^2\\ &\equiv (2x^2 - 3a^2 + 3xa)(2x^2 - 3a^2 - 3xa).\end{aligned}$$

Exercise 42.

Find the quadratic factors of

1. $x^4 + x^2 + 1$.
2. $x^4 - 3x^2 + 9$.
3. $x^4 - 13x^2 + 36$.
4. $x^4 - 6a^2x^2 + a^4$.
5. $x^4 - 11a^2x^2 + a^4$.
6. $x^4 + (4 - c^2)x^2y^2 + 4y^4$.
7. $(x+y)^4 + (x+y)^2 + 1$.
8. $9x^4 + 3x^2y^2 + 4y^4$.
9. $25x^4 - 44x^2y^2 + 16y^4$.
10. $4x^4 - 4x^2y^2 + 9y^4$.
11. $9x^4 - 12x^2y^2 + 16y^4$.
12. $16x^4 - x^2y^2 + y^4$.
13. $25x^4 - 29x^2y^2 + 4y^4$.
14. $x^4 - x^2y^2 + y^4$.
15. $36x^4 + 23x^2a^2 + 16a^4$.
16. $x^8 + x^4y^4 + y^8$.

150. Trinomials of the Form $x^2 + px + q$. As will be shown in § 186, any expression of this form can be resolved into two factors of the first degree in x by writing it as the difference of two squares. We give below a method of factoring the simpler cases by inspection.

By § 141, $\quad x^2 + (m+n)x + mn \equiv (x+m)(x+n)$;

hence $\quad x^2 + px + q \equiv (x+m)(x+n)$,

if $m + n = p$ and $mn = q$.

For many values of p and q, the values of m and n are whole numbers and can be readily found by inspection.

Example 1. Factor $x^2 + 7x + 12$.

Here $\quad x^2 + 7x + 12 \equiv (x+m)(x+n)$,

if $\quad m + n = 7$, and $mn = 12$. $\quad$ (1)

Since $mn = +12$, m and n have the same quality; hence, as $m + n = +7$, both m and n are $+$. Now the pairs of positive numbers whose product is 12 are 12 and 1, 6 and 2, and 4 and 3; and the sum of the last pair is 7.

Hence $\quad m = 4$, and $n = 3$ satisfy equations (1).

$$\therefore\ x^2 + 7x + 12 \equiv (x + 4)(x + 3).$$

Example 2. Factor $x^2 - 7x + 10$.

Here $\quad x^2 - 7x + 10 \equiv (x + m)(x + n)$,

if $\quad m + n = -7$, and $mn = 10$. $\quad$ (1)

Since $mn = +10$, m and n have the same quality, hence, as $m + n = -7$, both are $-$. Now pairs of negative numbers whose product is 10, are -10 and -1, -5 and -2; and the sum of the last pair is -7; hence $m = -5$, and $n = -2$ satisfy equations (1).

$$\therefore\ x^2 - 7x + 10 \equiv (x - 5)(x - 2).$$

Example 3. Factor $x^2 + 3x - 18$.

Here $\quad m + n = 3$, and $mn = -18$.

Since $mn = -18$, m and n are opposite in quality: hence, as $m + n = +3$, the positive number is numerically the larger. Now pairs of numbers whose product is -18, the larger numerically being $+$, are $+18$ and -1, $+9$ and -2, $+6$ and -3; and the sum of the last pair is $+3$. Hence $m = 6$, and $n = -3$,

and $\quad x^2 + 3x - 18 \equiv (x + 6)(x - 3)$.

Example 4. Factor $4a - a^2 + 21$.

$$4a - a^2 + 21 \equiv -(a^2 - 4a - 21)$$
$$\equiv -(a - 7)(a + 3) \equiv (7 - a)(a + 3).$$

Example 5. Factor $a^2x^2 + 5ax - 84$.

Here $\quad (ax)^2 + 5(ax) - 84 \equiv (ax + m)(ax + n)$,

if $\quad m + n = 5$, and $mn = -84$.

Hence $\quad m = 12$, and $n = -7$,

and $\quad a^2x^2 + 5ax - 84 \equiv (ax + 12)(ax - 7)$.

Example 6. Factor $x^2 - 32xy - 105y^2$.

Here $x^2 - 32xy - 105y^2 \equiv (x + m)(x + n)$

if $m + n = -32y$, and $mn = -105y$.

Hence $m = -35y$, and $n = 3y$,

and $x^2 - 32xy - 105y^2 \equiv (x - 35y)(x + 3y)$.

Exercise 43.

Factor

1. $x^2 + 4x + 3$.
2. $x^2 - 4x + 3$.
3. $x^2 + 9x + 20$.
4. $x^2 - 11x + 18$.
5. $x^2 - 8x + 15$.
6. $x^2 + 2x - 3$.
7. $x^2 + x - 6$.
8. $x^2 + 4x - 5$.
9. $x^2 + 2x - 35$.
10. $x^2 - 3x - 10$.
11. $x - x^2 + 6$.
12. $x^2 + 5x - 14$.
13. $x^2 + 18x + 72$.
14. $x - x^2 + 132$.
15. $x^2 - 5x - 84$.
16. $x^2 + 5x - 150$.
17. $x^2 - 25x + 150$.
18. $x^2 + 11x - 180$.
19. $x - x^2 + 156$.
20. $x^2 - 31x + 240$.
21. $x^2 - 34x + 288$.
22. $x^2 - 35x - 200$.
23. $x^2 - 17x - 200$.
24. $a^2x^2 - 21ax + 108$.
25. $a^2x^2 - 21ax + 80$.
26. $a^2x^2 + 21ax + 90$.
27. $a^2x^2 - 19ax + 78$.
28. $a^2x^2 + 30ax + 225$.
29. $a^2x^2 + 54ax + 729$.
30. $a^2x^2 - 38ax + 361$.
31. $x^2y^2 - 5xy - 24$.
32. $x^2 + 7xy - 60y^2$.
33. $x^2 + ax - 42a^2$.
34. $a^2 - ay - 210y^2$.
35. $x^2 + 18x - 115$.
36. $x^2 - 20xy - 96y^2$.
37. $x^2 - 26xy + 169y^2$.
38. $x^2 - 23xy + 132y^2$.
39. $x^2 + 9xy + 14y^2$.
40. $x^2 + 49xy + 600y^2$.

41.	$x^2+43xy+390y^2$.	**46.**	$x^4+13a^2x^2-300a^4$.
42.	$a^2-20abx+75b^2x^2$.	**47.**	$x^4-a^2x^2-462a^4$.
43.	$a^2-29ab+54b^2$.	**48.**	$x^4-a^2x^2-132a^4$.
44.	$130+31xy+x^2y^2$.	**49.**	$143-24xa+x^2a^2$.
45.	$a^2+12abx-28b^2x^2$.	**50.**	$216+35x+x^2$.

51. $65+8xy-x^2y^2$, or $-(x^2y^2-8xy-65)$.

52.	$110-x-x^2$.	**54.**	$380-x-x^2$.
53.	$98-7x-x^2$.	**55.**	$120-7ax-a^2x^2$.

151. Trinomials of the Form ax^2+bx+c.

$$a(ax^2+bx+c)\equiv(ax)^2+b(ax)+ac. \qquad (1)$$

Now when its factors involve only whole numbers, the second member of (1) can be readily factored by the method of § 150.

Example 1. Factor $3x^2-16x+5$.

$$\begin{aligned}3(3x^2-16x+5)&\equiv(3x)^2-16(3x)+15\\&\equiv(3x-15)(3x-1). \qquad \S\ 150\end{aligned}$$

$$\therefore\ 3x^2-16x+5\equiv(x-5)(3x-1).$$

Example 2. Factor $5x^2+32x-21$.

$$\begin{aligned}5(5x^2+32x-21)&\equiv(5x)^2+32(5x)-105\\&\equiv(5x+35)(5x-3).\end{aligned}$$

$$\therefore\ 5x^2+32x-21\equiv(x+7)(5x-3).$$

Example 3. Factor $3x^2-17xy+10y^2$.

$$\begin{aligned}3(3x^2-17xy+10y^2)&\equiv(3x)^2-17y(3x)+30y^2\\&\equiv(3x-15y)(3x-2y).\end{aligned}$$

$$\therefore\ 3x^2-17xy+10y^2\equiv(x-5y)(3x-2y).$$

Exercise 44.

Factor

1. $2x^2+3x+1$.
2. $3x^2+5x+2$.
3. $3x^2+10x+3$.
4. $3x^2+8x+4$.
5. $2x^2+7x+6$.
6. $2x^2+11x+5$.
7. $5x^2+11x+2$.
8. $2x^2+3x-2$.
9. $4x^2+11x-3$.
10. $2x^2+15x-8$.
11. $3x^2+7x-6$.
12. $2x^2+x-28$.
13. $3x^2+13x-30$.
14. $6x^2+7x-3$.
15. $3a^2x^2+23ax+14$.
16. $3a^2x^2+19ax-14$.
17. $6a^2x^2-31ax+35$.
18. $3x^2+41x+26$.
19. $4x^2+23x+15$.
20. $3x^2-13x+14$.
21. $2x^2-5xy-3y^2$.
22. $3x^2-17xy+10y^2$.
23. $12x^2-23xy+10y^2$.
24. $24x^2-29xy-4y^2$.
25. $7x^2-33xy-54y^2$.
26. $132x^2+x-1$.
27. $12x^2+50x-50$.
28. $15x^4y-4x^3y^2-4x^2y^3$.
29. $ax^2+(a+b)x+b$.
30. $ax^2+(a-b)x-b$.

152. Binomials of the Form $x^n - a^n$. By § 142,

$x^n - a^n$ *is divisible by* $x - a$ *always.*

The other factor can be written out by § 142.

Example 1. Factor $343 - 8a^3$.

$$\begin{aligned}343 - 8a^3 &\equiv (7)^3 - (2a)^3\\ &\equiv (7-2a)[7^2 + 7(2a) + (2a)^2]\\ &\equiv (7-2a)(49 + 14a + 4a^2).\end{aligned}$$

Example 2. Factor $2a^5 - 64b^5$.

$$\begin{aligned}2a^5 - 64b^5 &\equiv 2[a^5 - (2b)^5]\\ &\equiv 2(a-2b)[a^4 + a^3(2b) + a^2(2b)^2 + a(2b)^3 + (2b)^4]\\ &\equiv 2(a-2b)(a^4 + 2a^3b + 4a^2b^2 + 8ab^3 + 16b^4).\end{aligned}$$

When n is even, $x^n - a^n$ should be factored as the difference between two squares.

$$\text{Thus}\quad x^4 - a^4 \equiv (x^2 + a^2)(x^2 - a^2)$$
$$\equiv (x^2 + a^2)(x + a)(x - a).$$

153. Factors of $x^n + a^n$ when n is odd. By § 144, $x^n + a^n$ *is divisible by* $x + a$ *when* n *is odd.*

EXAMPLE 1. Factor $8a^3 + 27b^3$.

$$8a^3 + 27b^3 \equiv (2a)^3 + (3b)^3$$
$$\equiv (2a + 3b)[(2a)^2 - (2a)(3b) + (3b)^2]$$
$$\equiv (2a + 3b)(4a^2 - 6ab + 9b^2).$$

EXAMPLE 2. Factor $x^6 - a^6$.

$$x^6 - a^6 \equiv (x^3 + a^3)(x^3 - a^3)$$
$$\equiv (x + a)(x^2 - ax + a^2)(x - a)(x^2 + ax + a).$$

Exercise 45.

Factor

1. $a^3 - b^3$.
2. $a^3 + b^3$.
3. $a^5 - b^5$.
4. $a^5 + b^5$.
5. $a^7 + b^7$.
6. $a^7 - b^7$.
7. $x^3 + 1$.
8. $x^3 + 27$.
9. $125 - x^3$.
10. $x^3 - 8b^3$.
11. $27x^3 + 1$.
12. $216 - x^3$.
13. $x^3b^3 - \frac{1}{8}a^3$.
14. $343 - 8x^3$.
15. $8x^3 - 27a^6$.
16. $40a^3 - 135b^3$.
17. $27a^3 + 64x^3$.
18. $y^5 - 1$.
19. $x^4 - 16b^4$.
20. $16x^4 - 81a^4$.
21. $81a^4x^4 - 1$.
22. $16 - 81a^4b^4$.
23. $a^6 - 729b^6$.
24. $x^6 - 1$.
25. $x^6 - 64$.
26. $x^6y^6 - a^6b^6$.
27. $64x^6 - 729b^6$.
28. $32x^5 - 1$.
29. $a^5b^5 - 243$.
30. $1024a^5 - 32b^5$.

154. **Factors of $x^n + a^n$ when n is even.**

(i.)
$$\begin{aligned} x^4 + a^4 &= x^4 + a^4 + 2a^2x^2 - 2a^2x^2 \\ &= (x^2 + a^2)^2 - 2a^2x^2 \\ &= (x^2 + a^2 + ax\sqrt{2})(x^2 + a^2 - ax\sqrt{2}). \end{aligned}$$

These factors are rational with respect to a and x.

The same method may be employed whenever n is a multiple of 4, as when n is 8, 12, 16, etc.

(ii.)
$$\begin{aligned} x^6 + a^6 &= (x^2)^3 + (a^2)^3 \\ &= (x^2 + a^2)(x^4 - x^2a^2 + a^4). \\ x^4 - x^2a^2 + a^4 &= x^4 + 2a^2x^2 + a^4 - 3a^2x^2 \\ &= (x^2 + a^2)^2 - 3a^2x^2 \\ &= (x^2 + a^2 + ax\sqrt{3})(x^2 + a^2 - ax\sqrt{3}). \\ \therefore\ x^6 + a^6 &= (x^2 + a^2)(x^2 + a^2 + ax\sqrt{3})(x^2 + a^2 - ax\sqrt{3}). \end{aligned}$$

The same method may be employed whenever one of the two factors of n is odd, as when n is 10, 12, 14, etc.

Exercise 46.

Factor

1. $x^4 + 1$.
2. $x^6 + 1$.
3. $16x^4 + a^4$.
4. $x^6 + 64$.
5. $x^8 - a^8$.
6. $1 - x^8$.
7. $x^8 + a^8$.
8. $x^{10} + a^{10}$.
9. $x^{12} + a^{12}$.

155. The terms of a polynomial can often be grouped so as to show a common factor.

Example 1. Factor $ax + by + bx + ay$.

Combining like terms, we have

$$\begin{aligned} ax + by + bx + ay &\equiv (a + b)x + (a + b)y \\ &\equiv (a + b)(x + y). \end{aligned}$$

Example 2. Factor $x^3 - 3x^2 - x + 3$.

$$\begin{aligned} x^3 - 3x^2 - a + 3 &\equiv (x - 3)x^2 - (x - 3) \\ &\equiv (x - 3)(x^2 - 1) = (x - 3)(x - 1)(x + 1) \end{aligned}$$

Example 3. Factor $ax^3 - x - a + 1$.

$$\begin{aligned} ax^3 - x - a + 1 &\equiv a(x^3 - 1) - (x - 1) \\ &\equiv (x - 1)[a(x^2 + x + 1) - 1] \\ &\equiv (x - 1)(ax^2 + ax + a - 1). \end{aligned}$$

Exercise 47.

Factor

1. $a^2 + ab + ac + bc$.
2. $a^2c^2 + acd + abc + bd$.
3. $a^2 + 3a + ac + 3c$.
4. $mx - my - nx + ny$.
5. $3ax - bx - 3ay + by$.
6. $6x^2 + 3xy - 2ax - ay$.
7. $ax^2 - 3bxy - axy + 3by^2$.
8. $2ax^2 + 3axy - 2bxy - 3by^2$.
9. $amx^2 + bmxy - anxy - bny^2$.
10. $ax - bx + by + cy - cx - ay$.
11. $a^2x + abx + ac + aby + b^2y + bc$.
12. $x^3 + x^2 - 4x - 4$.
13. $5x^3 - x^2 - 5x + 1$.
14. $ax^3 + bx^3 + a + b$.
15. $ax^2 + by^2 + (a + b)xy$.
16. $a^2b^2 + a^2 + b^2 + 1$.
17. $a^4 + a^2b^2 - b^2c^2 - c^4$.
18. $a^2 - a - c^2 + c$.
19. $a^2 - b^2 - (a - b)^2$.
20. $a^2 - b^2 + bc - ca$.
21. $ax^3 + x^3 + a + 1$.
22. $x^3 - 5x^2 + x - 5$.
23. $2x^3 - 3x^2 - 2x + 3$.
24. $x^3 + bx^2 - a^2x - a^2b$.
25. $a^2b^2 - a^2 - b^2 + 1$.
26. $bx^3 + ax^2 + bx + a$.
27. $x^2 - y^2 + xz - yz$.
28. $1 + bx - (a^2 + ab)x^2$.
29. $a^2c^2 + acd + abc + bd$.
30. $ac + bd - ad - bc$.
31. $ac^2 + bd^2 - ad^2 - bc^2$.
32. $a^2x - b^2x + a^2y - b^2y$.
33. $a^3x^2 - c^3x^2 - a^3y^2 + c^3y^2$.

34. $a^2x^6 - a^2y^6 - b^2x^6 + b^2y^6.$

35. $acx^2 - bcx + adx - bd.$

36. $c^5d^3 - c^2 - a^2c^3d^3 + a^2.$

37. $1 - abx^3 + (b - a^2)x^2.$

38. $a^2 - b^2 + c^2 - d^2 - 2(ac - bd).$

39. $4a^2b^2 - (a^2 + b^2 - c^2)^2.$

40. $(a^2 - b^2 + c^2 - d^2)^2 - (2ac - 2bd)^2.$

41. $x^4 + x^3y + xz^3 + yz^3.$

42. $x(x + z) - y(y + z).$

43. $x^4 - 14x^2y^2 + y^4.$

44. $x^2y^2 - x^2z^2 - y^2z^2 + z^4.$

45. $1 - 2ax - (c - a^2)x^2 + acx^3.$

46. $a^2(b - c) + b^2(c - a) + c^2(a - b).$

47. $b^2c + bc^2 + c^2a + ca^2 + a^2b + ab^2 + 2abc.$

48. $ax(y^3 + b^3) + by(bx^2 + a^2y).$

49. $2x^3 - 4x^2y - x^2z + 2xy^2 + 2xyz - y^2z.$

50. $(x^2 + 4x + 8)^2 + 3x(x^2 + 4x + 8) + 2x^2.$

CHAPTER XII.

COMMON FACTORS AND MULTIPLES.

156. A **Common Factor** of two or more expressions is an expression which will exactly divide each of them.

Thus $a - x$ is a common factor of $b(a - x)$ and $a^2 - x^2$.

157. Two expressions are said to be **Prime** to each other, when they have no common factor except unity.

158. The **Highest Common Factor** of two or more algebraic expressions is the expression of highest degree which will exactly divide each of them.

Thus $x^2y^2z^3$ is the Highest Common Factor of the expressions $x^3y^2z^4$ and $x^2y^4z^3$.

For the words Highest Common Factor, we shall frequently use the abbreviation H. C. F.

If the expressions have numerical coefficients, the Greatest C. F. of these coefficients is prefixed as a coefficient to the algebraic H. C. F.

Thus $10\,x^3yz$ is the H. C. F. of $20\,x^4yz$ and $30\,x^3y^2z^3$.

159. To obtain the H. C. F. of two or more expressions:

Resolve each expression into its factors. Find the product of all their common factors, each being raised to the lowest power in which it occurs in any of the expressions.

Example 1. Find the H. C. F. of $6a^2b^3c^3d^2$ and $4a^3c^5d$.

$$6a^2b^3c^3d^2 = 3 \cdot 2 \cdot a^2 \cdot b^3 \cdot c^3 \cdot d^2.$$
$$4a^3c^5d = 2 \cdot 2 \cdot a^3 \cdot c^5 \cdot d.$$

The only common factors are 2, a^2, c^3, and d; hence

$$\text{the H.C.F.} = 2 \cdot a^2 \cdot c^3 \cdot d, \text{ or } 2a^2c^3d.$$

Example 2. Find the H. C. F. of $a^4b^2 - a^2b^4$ and $a^4b^3 + a^3b^4$.

$$a^4b^2 - a^2b^4 = a^2b^2(a+b)(a-b);$$
$$a^4b^3 + a^3b^4 = a^3b^3(a+b).$$
$$\therefore \text{ H.C.F.} = a^2b^2(a+b).$$

Example 3. Find the H. C. F. of $3a^4 + 15a^3b - 72a^2b^2$ and $6a^3 - 30a^2b + 36ab^2$.

$$3a^4 + 15a^3b - 72a^2b^2 = 3a^2(a+8b)(a-3b);$$
$$6a^3 - 30a^2b + 36ab^2 = 6a(a-2b)(a-3b).$$
$$\therefore \text{ H.C.F.} = 3a(a-3b).$$

Exercise 48.

Find the H. C. F. of

1. $9ab^3$ and $6a^2b$.
2. $24a^3b^3x^4$ and $60a^2b^4x^6$.
3. $42axy^2z^3$ and $77b^3y^4$.
4. ab^3, a^2bc, and abc^2.
5. $3x^2yz^3$, $15xy^3z^2$, and $10x^2y^2$.
6. $3(x-y)^3$ and $6(x-y)^4$.
7. $(x-a)^3(x+b)^3$ and $(x-a)^2(x+b)^4$.
8. $6a^2(x-y)^3$ and $9a^3(x-y)^5$.
9. $a^2x^3 + 2a^3x^2$ and $a^2x^4 - 4a^4x^2$.
10. $a^6x^2 - 4a^4x^4$ and $a^6x^2 - 16a^2x^6$.
11. $x^2 + 3x + 2$ and $x^2 + 6x + 8$.
12. $x^3 + 3x^2y + 2xy^2$ and $x^4 + 6x^3y + 8x^2y^2$.

$$\begin{aligned} x^3 + 3x^2y + 2xy^2 &= (x^3 + 2x^2y) + (x^2y + 2xy^2) \\ &= x^2(x+2y) + xy(x+2y) \\ &= (x^2 + xy)(x+2y) \\ &= x(x+y)(x+2y). \end{aligned}$$

13. $3x^2-4x+1$ and $4x^2-5x+1$.

14. $3a^2-4ab+b^2$ and $4a^4-5a^3b+a^2b^2$.

15. a^3-a^2x, a^3-ax^2, and a^4-ax^3.

16. x^2-1, x^3+1, and x^2-2x-3.

17. $2x^2-7x+3$ and $3x^2-7x-6$.

18. $12x^2+x-1$ and $15x^2+8x+1$.

19. $2x^2-x-1$ and $3x^2-x-2$.

20. $c^2x^2-d^2$ and $acx^2-bcx+adx-bd$.

21. x^5-xy^2 and $x^3+x^2y+xy+y^2$.

22. $a^3x-a^2bx-6ab^2x$ and $a^2bx^2-4ab^2x^2+3b^3x^2$.

23. $2x^2+9x+4$, $2x^2+11x+5$, and $2x^2-3x-2$.

160. The Polynomial Factor of the H. C. F. of two expressions can always be found by a process analogous to that employed in arithmetic to find the greatest common divisor of two numbers.

RULE. *Arrange the two expressions in descending powers of some common letter, and divide the expression which is of the highest degree in the common letter by the other; if both expressions are of the same degree, either may be used as the divisor.*

Take the remainder, if any, after the first division for a new divisor, and the former divisor as dividend. Continue this process until the last remainder is zero, or does not contain the letter of arrangement. If the last remainder is zero, the last divisor is the H. C. F. sought; if the last remainder is not zero, the two expressions have no common polynomial factor in the letter of arrangement.

Example 1. Find the H. C. F. of $x^2 + x^3 - 2$ and $x^3 + 2x^2 - 3$.

First arrange both expressions in descending powers of x.

$$\begin{array}{l}
x^3 + x^2 - 2)\, x^3 + 2x^2 - 3\, (1 \\
\qquad\qquad\quad \underline{x^3 + \;\; x^2 - 2} \\
\qquad\qquad\qquad\qquad x^2 - 1)\, x^3 + x^2 - 2\, (x + 1 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^3 - x \qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\; x^2 + x - 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\; \underline{x^2 \qquad - 1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x - 1)\, x^2 - 1\, (x + 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; \underline{x^2 - x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x - 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x - 1}
\end{array}$$

Here the last remainder is zero, and the last divisor is $x - 1$; hence the H. C. F. sought is $x - 1$.

Before employing the method given above, *all monomial* factors should be removed from the given polynomials, and the monomial factor of the H. C. F. found by inspection. Any monomial factor also should be removed from each remainder before it is used as a divisor; for any such monomial factor cannot be a factor of the H. C. F. sought.

Example 2. Find the H. C. F. of

$$3c^2x^4 + 3c^2x^3 - 6c^2x \text{ and } 6cx^5 + 12cx^4 - 18cx^2.$$

Now $\qquad 3c^2x^4 + 3c^2x^3 - 6c^2x = 3c^2x(x^3 + x^2 - 2)$,

and $\qquad 6cx^5 + 12cx^4 - 18cx^2 = 6cx^2(x^3 + 2x^2 - 3)$.

The H. C. F. of the monomial factors is $3cx$; and by Example 1, the H. C. F. of the trinomial factors is $x - 1$.

Hence the H. C. F. of the given expressions is $3cx(x - 1)$.

161. *The H. C. F. of two expressions will not be changed if either expression be multiplied or divided by a factor that is not a factor of the other.*

For this will not change either the number or the value of their *common* factors.

Example 1. Find the H. C. F. of

$$2ax^3 + 8ax^2 - 16ax + 48a \text{ and } 4a^2x^4 - 4a^2x^3 + 32a^2x - 32a^2.$$

$$4a^2x^4 - 4a^2x^3 + 32a^2x - 32a^2 = 4a^2(x^4 - x^3 + 8x - 8),$$

and $$2ax^3 + 8ax^2 - 16ax + 48a = 2a(x^3 + 4x^2 - 8x + 24).$$

The H. C. F. of the monomial factors is $2a$.

To find the H. C. F. of the polynomial factors we proceed as in § 160.

$$\begin{array}{r l}
x^3 + 4x^2 - 8x + 24) & x^4 - x^3 + 8x - 8\ (x - 5 \\
 & \underline{x^4 + 4x^3 - 8x^2 + 24x} \\
 & \quad -5x^3 + 8x^2 - 16x - 8 \\
 & \quad \underline{-5x^3 - 20x^2 + 40x - 120} \\
 & \qquad\quad 28x^2 - 56x + 112 = 28(x^2 - 2x + 4)
\end{array}$$

Rejecting the monomial factor 28, we continue the process with $x^2 - 2x + 4$ as the new divisor.

$$\begin{array}{r l}
x^2 - 2x + 4) & x^3 + 4x^2 - 8x + 24\ (x + 6 \\
 & \underline{x^3 - 2x^2 + 4x} \\
 & \qquad 6x^2 - 12x + 24 \\
 & \qquad \underline{6x^2 - 12x + 24}
\end{array}$$

Hence the H. C. F. of the polynomial factors is $x^2 - 2x + 4$.

Therefore the H. C. F. sought is $2a(x^2 - 2x + 4)$.

Example 2. Find the H. C. F. of

$$2x^2 - 5x + 2 \text{ and } x^3 + 4x^2 - 4x - 16.$$

Since 2 is not a factor of $2x^2 - 5x + 2$, we may by § 161 multiply $x^3 + 4x^2 - 4x - 16$ by 2, and thus avoid the inconvenience of fractions.

The work may be written as below:

$$\begin{array}{ll}
 & 2x^2-5x+2)\;x^3+4x^2-4x-16 \\
\text{Multiply by 2,} & \underline{\qquad\qquad\quad 2\qquad\qquad\qquad} \\
 & 2x^3+8x^2-8x-32\;(x \\
 & \underline{2x^3-5x^2+2x} \\
 & \qquad 13x^2-10x-32 \\
\text{Multiply by 2,} & \qquad \underline{\;2\qquad\qquad\qquad} \\
 & \qquad 26x^2-20x-64\;(13 \\
 & \qquad \underline{26x^2-65x+26} \\
\text{Divide by 45,} & \qquad\quad 45)\;\underline{45x-90} \\
 & \qquad\qquad\quad x-2)\;2x^2-5x+2\;(2x-1 \\
 & \qquad\qquad\qquad\qquad \underline{2x^2-4x} \\
 & \qquad\qquad\qquad\qquad\quad -x+2 \\
 & \qquad\qquad\qquad\qquad\quad \underline{-x+2}
\end{array}$$

Hence the H. C. F. is $x-2$.

Example 3. Find the H. C. F. of

$$2x^3+x^2-x-2 \text{ and } 3x^3-2x^2+x-2.$$

Multiply the last expression by 2.

$$\begin{array}{l}
2x^3+x^2-x-2)\;6x^3-4x^2+2x-4\;(3 \\
\qquad\qquad\qquad\quad \underline{6x^3+3x^2-3x-6} \\
\qquad\qquad\qquad\qquad -7x^2+5x+2
\end{array}$$

Multiply the first divisor by 7.

$$\begin{array}{ll}
 & -7x^2+5x+2)\;14x^3+7x^2-7x-14\;(-2x \\
 & \qquad\qquad\qquad \underline{14x^3-10x^2-4x} \\
 & \qquad\qquad\qquad\qquad 17x^2-3x-14 \\
\text{Multiply by 7,} & \qquad\qquad\qquad\qquad \underline{\;7\qquad\qquad\qquad} \\
 & \qquad\qquad\qquad\qquad 119x^2-21x-98\;(-17 \\
 & \qquad\qquad\qquad\qquad \underline{119x^2-85x-34} \\
\text{Divide by 64,} & \qquad\qquad\qquad\qquad\quad 64)\;\underline{64x-64} \\
 & \qquad\qquad\qquad\qquad\qquad x-1)\;-7x^2+5x+2\;(-7x-2 \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\; \underline{-7x^2+7x} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad -2x+2 \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-2x+2}
\end{array}$$

Hence the H. C. F. sought is $x-1$.

Note. Before proving the rule for finding the H. C. F. of two polynomials, the student should make himself familiar with it by solving some of the examples in Exercise 49.

The method of finding the H. C. F. which is given above is based upon the following principle:

162. *If* $A = BQ + R$, *then the H. C. F. of* B *and* R *is the same as the H. C. F. of* A *and* B.

For, if $A = BQ + R$,

then $R = A - BQ$.

Now every factor common to B and R is by § 146 a factor of $BQ + R$, or A; hence every factor common to B and R is common to A and B.

Again, every factor common to A and B is a factor of $A - BQ$, or R; hence every factor common to A and B is common to B and R.

Hence, if $A = BQ + R$, the H. C. F. of B and R is the H. C. F. of A and B.

163. Proof of Rule for finding H. C. F. Let A and B denote any two polynomials arranged according to the descending powers of some common letter, the degree of B not being higher than that of A.

Divide A by B and let the quotient be Q' and the remainder R'. Divide B by R' and let the quotient be Q'', and the remainder R''; and so on.

Then
$$A = BQ' + R', \quad (1)$$
$$B = R'Q'' + R''. \quad (2)$$

By § 162, from (1) the H. C. F. of B and R' is the H. C. F. of A and B; from (2) the H. C. F. of R' and R'' is the H. C. F. of B and R', that is, the H. C. F. of A and B; and so on.

Hence, in the process of finding the H. C. F. of two expressions, ***the H. C. F. of any divisor and the corresponding dividend is the H. C. F. sought.***

Now if at any stage of the process there is no remainder, the last divisor is a factor of the corresponding dividend; therefore it is evidently the H. C. F. of itself and that dividend. Hence it is the H. C. F. sought.

Again, each remainder is evidently of a lower degree than the preceding one. Hence, unless at some stage of the process the remainder is zero, we must come at last to a remainder which does not contain the letter of arrangement. In this case the given expressions have no common polynomial factor containing that letter; for by § 162 this last remainder contains all the polynomial factors common to the given expressions.

164. Any factor common to three or more expressions must be a factor of the H. C. F. of any two of them.

Hence, to find the H. C. F. of three expressions, we may first find the H. C. F. of any two of them, and then find the H. C. F. of this result and the third.

Example. Find the H. C. F. of

$$x^3 + x^2 - x - 1,\ x^3 + 3x^2 - x - 3, \text{ and } x^3 + x^2 - 2.$$

The H. C. F. of the first two expressions is $x^2 - 1$. The H. C. F. of $x^2 - 1$ and $x^3 + x^2 - 2$ is $x - 1$. Hence the H. C. F. sought is $x - 1$.

Exercise 49.

Find the H. C. F. of

1. $x^2 - 5x + 4,\ x^3 - 5x^2 + 4.$
2. $x^2 - 5xy + 4y^2,\ x^4 - 5x^3y + 4xy^3.$
3. $2x^2 - 5x + 2,\ 4x^3 + 12x^2 - x - 3.$

4. $x^3-5x^2-99x+40,\ x^3-6x^2-86x+35.$
5. $x^3+2x^2-8x-16,\ x^3+3x^2-8x-24.$
6. $x^3-x^2-5x-3,\ x^3-4x^2-11x-6.$
7. $x^3+3x^2-8x-24,\ x^3+3x^2-3x-9.$
8. $x^6+3x^4y-10x^2y^2,\ x^4-3x^2y+2y^2.$
9. $2a^2-5a+2,\ 2a^3-3a^2-8a+12.$
10. $2b^2-5b+2,\ 12b^3-8b^2-3b+2.$
11. $a^3-5a^2x+7ax^2-3x^3,\ a^3-3ax^2+2x^3.$
12. $x^4-2x^3-4x-7,\ x^4+x^3-3x^2-x+2.$
13. $x^3-3a^2x-2a^3,\ x^3-ax^2-4a^3.$
14. $2x^3+4x^2-7x-14,\ 6x^3-10x^2-21x+35.$
15. $2x^4-2x^3+x^2+3x-6,\ 4x^4-2x^3+3x-9.$
16. $3x^3+x^2+x-2,\ 2x^3-x^2-x-3.$
17. $3x^3-3ax^2+2a^2x-2a^3,\ 3x^3+12ax^2+2a^2x+8a^3.$
18. $3x^3-3x^2y+xy^2-y^3,\ 4x^2y-5xy^2+y^3.$
19. $12x^2-15xy+3y^2,\ 6x^3-6x^2y+2xy^2-2y^3.$
20. $10x^3+25ax^2-5a^3,\ 4x^3+9ax^2-2a^2x-a^3.$
21. $6a^3+13a^2x-9ax^2-10x^3,\ 9a^3+12a^2x-11ax^2-10x^3.$
22. $2x^4+9x^3+14x+3,\ 2+9x+14x^3+3x^4.$
23. $3x^4+5x^3-7x^2+2x+2,\ 2x^4+3x^3-2x^2+12x+5.$
24. $2x^3-11x^2+11x+4,\ 2x^4-3x^3+7x^2-12x-4.$
25. $2x^4+4x^3+3x^2-2x-2,\ 3x^4+6x^3+7x^2+2x+2.$
26. $x^2-9x-10,\ x^2-7x-30,\ x^2-11x+10.$
27. $x^2+x-6,\ x^3-2x^2-x+2,\ x^3+3x^2-6x-8.$
28. $x^3+7x^2+5x-1,\ x^2+3x-3x^3-1,\ 3x^3+5x^2+x-1.$

LOWEST COMMON MULTIPLE.

165. A **Common Multiple** of two or more expressions is an expression that is exactly divisible by each of them.

The **Lowest Common Multiple** (abbreviated L. C. M.) of two or more expressions is the expression of lowest degree that is exactly divisible by each of them.

Thus $6a^2b^3$ is the L. C. M. of $3a^2b$ and $2ab^3$.

166. To find the L. C. M. of two or more expressions that can be resolved into their prime factors, we have the following rule:

Resolve the expressions into their prime factors.

Take the product of all the different factors, each being raised to the highest power in which it occurs in any one of the expressions.

For evidently the L. C. M. must contain each factor the greatest number of times that it occurs in any one of the expressions.

Example 1. Find the L. C. M. of $42axy^2z^3$ and $77b^3y^4$.

$$42axy^2z^3 = 6 \cdot 7 \cdot a \cdot x \cdot y^2 \cdot z^3.$$

$$77b^3y^4 = 11 \cdot 7 \cdot b^3 \cdot y^4.$$

$$\therefore \text{ L. C. M.} = 6 \cdot 7 \cdot 11 \cdot a \cdot b^3 \cdot x \cdot y^4 \cdot z^3$$

$$= 462ab^3xy^4z^3.$$

Example 2. Find the L. C. M. of $x^2 + 7x + 12$, $x^2 + 6x + 8$, and $5x^2 + 20x + 20$.

$$x^2 + 7x + 12 = (x + 3)(x + 4);$$

$$x^2 + 6x + 8 = (x + 2)(x + 4);$$

$$5x^2 + 20x + 20 = 5(x + 2)^2.$$

$$\therefore \text{ L. C. M.} = 5(x + 2)^2(x + 3)(x + 4).$$

Exercise 50.

Find the L. C. M. of

1. $4x^3y,\ 10xy^3$.
2. $24a^3b^3x^4,\ 60a^2b^4x^6$.
3. $9a^2b^3x^4y^5,\ 8x^3y^6$.
4. $x^2,\ x^2-3x$.
5. $21x^3,\ 7x^2(x+1)$.
6. $6x^2-2x,\ 9x^2-3x$.
7. $x^2+2x,\ x^2+3x+2$.
8. $x^2-5x+4,\ x^2-6x+8$.
9. $x^2+4x+4,\ x^2+5x+6$.
10. $x^2-x-6,\ x^2+x-2,\ x^2-4x+3$.
11. $x^2+x-20,\ x^2-10x+24,\ x^2-x-30$.
12. $x^2+x-42,\ x^2-11x+30,\ x^2+2x-35$.
13. $2x^2+3x+1,\ 2x^2+5x+2,\ x^2+3x+2$.
14. $5x^2+11x+2,\ 5x^2+16x+3,\ x^2+5x+6$.
15. $x^2-7xy+12y^2,\ x^2-6xy+8y^2,\ x^2-5xy+6y^2$.
16. $2x^2+3x-2,\ 2x^2+15x-8,\ x^2+10x+16$.
17. $8x^2-38xy+35y^2,\ 4x^2-xy-5y^2,\ 2x^2-5xy-7y^2$.

167. To find the L. C. M. of two expressions that cannot be factored by inspection, we have the following rule:

Find the H. C. F. of the two expressions.

Divide one of the expressions by their H. C. F. and multiply the quotient by the other.

For, let A and B denote any two algebraic expressions, H their H. C. F., and L their L. C. M. Now L evidently contains all the factors of A, and in addition all the factors of B that are not in A, that is, the factors of $B \div H$.

Hence $$L = A \times (B \div H). \quad (1)$$

Example. Find the L. C. M. of $x^3 + x^2 - 2$ and $x^3 + 2x^2 - 3$. Here the H. C. F. is $x - 1$.

$$(x^3 + x^2 - 2) \div (x - 1) = x^2 + 2x + 2.$$

$$\therefore \text{ L. C. M.} = (x^3 + 2x^2 - 3)(x^2 + 2x + 2).$$

Exercise 51.

Find the L. C. M. of

1. $2x^2 + 3x - 20,\ 6x^3 - 25x^2 + 21x + 10.$
2. $x^2 - 15x + 36,\ x^3 - 3x^2 - 2x + 6.$
3. $9x^3 - x - 2,\ 3x^3 - 10x^2 - 7x - 4.$
4. $x^3 + x^2 - 4x - 4,\ x^3 + 6x^2 + 11x + 6.$
5. $x^3 - x^2 - 7x + 15,\ x^3 + x^2 - 3x + 9.$
6. $x^3 - x^2 + x + 3,\ x^4 + x^3 - 3x^2 - x + 2.$
7. $x^4 - x^3 + 8x - 8,\ x^3 + 4x^2 - 8x + 24.$
8. $6x^3 + x^2 - 5x - 2,\ 6x^3 + 5x^2 - 3x - 2.$
9. $4x^3 - 10x^2 + 4x + 2,\ 3x^4 - 2x^3 - 3x + 2.$
10. $x^3 - 9x^2 + 26x - 24,\ x^3 - 12x^2 + 47x - 60.$
11. $x^3 - ax^2 - a^2x + a^3,\ x^3 + ax^2 - a^2x - a^3.$
12. Prove that the product of any two expressions is equal to the product of their H. C. F. and L. C. M.

CHAPTER XIII.

FRACTIONS.

168. A fraction is said to be in its *lowest terms* when its dividend and divisor have no common factor.

169. To reduce a fraction to its lowest terms,

Divide its dividend and divisor by all their common factors, or by their H. C. F.

For by § 83, we thus obtain an equal fraction of which the dividend and divisor have no common factors.

Example 1. Reduce $\frac{ax - a^2}{a^2 - x^2}$ to its lowest terms.

$$\frac{ax - a^2}{a^2 - x^2} = \frac{-a(a-x)}{(a+x)(a-x)}$$

$$= \frac{-a}{a+x} = -\frac{a}{a+x}.$$

Example 2. Reduce $\frac{x^2 - 7x + 10}{x^2 - 5x + 6}$ to its lowest terms.

$$\frac{x^2 - 7x + 10}{x^2 - 5x + 6} = \frac{(x-5)(x-2)}{(x-3)(x-2)} = \frac{x-5}{x-3}.$$

Example 3. Reduce $\frac{3x^3 - 13x^2 + 23x - 21}{15x^3 - 38x^2 - 2x + 21}$ to its lowest terms.

The H. C. F. of the dividend and divisor is found, by § 160, to be $3x - 7$. Dividing both the dividend and divisor by their H.C.F., we obtain

$$\frac{3x^3 - 13x^2 + 23x - 21}{15x^3 - 38x^2 - 2x + 21} = \frac{x^2 - 2x + 3}{5x^2 - x - 3}.$$

Exercise 52.

Reduce to its lowest terms:

1. $\dfrac{-x^3y}{-x^2y^3}$.

2. $-\dfrac{-2a^2bc^2}{4a^4b^3c}$.

3. $-\dfrac{x^4y^5z^2}{-x^6y^2z^2}$.

4. $\dfrac{12x^7y^2z^{10}}{16x^2y^2z^2}$.

5. $\dfrac{-15a^3b^2c^7x^5}{-25a^2b^4c^3x^8}$.

6. $\dfrac{125ab^2c^3d^4}{150a^4b^3c^2d}$.

7. $\dfrac{3a^5x^3yz^4}{-5ab^4xy^3z}$.

8. $\dfrac{5a^3b^4c^5xy^2}{-7b^4c^3xy^3}$.

9. $\dfrac{3a^3bc^4x^5y^2z}{4a^2b^3cx^3y^4z^3}$.

10. $\dfrac{a^2-ab}{a^2+ab}$.

11. $\dfrac{x^2+ax}{x^2-a^2}$.

12. $\dfrac{x^2-x^2y^2}{(x+xy)^2}$.

13. $\dfrac{x^2+2x}{x^2-4}$.

14. $\dfrac{4x-16}{x^2-16}$.

15. $\dfrac{2x^3-4x^4}{x^2-4x^4}$.

16. $\dfrac{x-2}{4-x^2}$.

17. $\dfrac{a-3}{9-a^2}$.

18. $\dfrac{x^4-a^2}{a^2-ax^2}$.

19. $\dfrac{x^2-1}{x^3-1}$.

20. $\dfrac{x^4-a^4}{x^6-a^6}$.

21. $\dfrac{15a^2-5ax}{x^2-9a^2}$.

22. $\dfrac{a^2-2ax+x^2}{x^2-a^2}$.

23. $\dfrac{a^4+2a^2b^2+b^4}{b^4-a^4}$.

24. $\dfrac{1-5a+6a^2}{1-7a+12a^2}$.

25. $\dfrac{x^2-9x+20}{x^2+6x-55}$.

26. $\dfrac{1-9y^2+20y^4}{1+6y^2-55y^4}$.

27. $\dfrac{a^4-b^4}{(a^3-b^3)(a+b)}$.

30. $\dfrac{4x^3+3ax^2+a^3}{x^4+ax^3+a^3x+a^4}$.

28. $\dfrac{x^2-8xy+7y^2}{x^2-3xy-28y^2}$.

31. $\dfrac{4x^3-10x^2+4x+2}{3x^4-2x^3-3x+2}$.

29. $\dfrac{(a^3-x^3)(a+x)}{(a^3+x^3)(a-x)}$.

32. $\dfrac{6x^3+x^2-5x-2}{6x^3+5x^2-3x-2}$.

33. $\dfrac{4x^4+11x^2+25}{4x^4-9x^2+30x-25}$.

34. $\dfrac{x^4-20x^2-15x+4}{x^4+9x^3+19x^2-9x-20}$.

170. To reduce a fraction to an integral or mixed expression,

Perform the indicated operation of division.

Sometimes the division may be performed by inspection, thus

$$\frac{x^3+a^3}{x+a}=x^2-ax+a^2. \tag{1}$$

$$\frac{x^3-1}{x+1}=\frac{(x^3+1)-2}{x+1}=x^2-x+1-\frac{2}{x+1}. \tag{2}$$

$$\frac{3x-2}{x+5}=\frac{3(x+5)-17}{x+5}=3-\frac{17}{x+5}. \tag{3}$$

In (2) and (3) the dividend is so modified that the part involving x is exactly divisible by the divisor. The principle involved is the distributive law of division.

Example. Reduce $\dfrac{4x^2-3x-54}{x-4}$ to a mixed expression.

By division, we obtain the quotient $4x+13$, and the remainder -2.

$$\therefore\ \frac{4x^2-3x-54}{x-4}=4x+13-\frac{2}{x-4}.$$

Exercise 53.

Reduce to an integral or mixed expression:

1. $\dfrac{5x^2-20x-15}{5x}$.
2. $\dfrac{x+7}{x+2}$.
3. $\dfrac{x^2}{x+3}$.
4. $\dfrac{5x+7}{x-3}$.
5. $\dfrac{x^2+a^2}{x+a}$.
6. $\dfrac{x^2+a^2}{x-a}$.
7. $\dfrac{x^3-a^3}{x+a}$.
8. $\dfrac{x^4+16}{x+2}$.
9. $\dfrac{x^4+a^4}{x-a}$.
10. $\dfrac{2x^2-7x-1}{x-3}$.
11. $\dfrac{x^2-3x}{x-2}$.
12. $\dfrac{3x^2+2x+1}{x+4}$.
13. $\dfrac{4a^2+6ab+9b^2}{2a-3b}$.
14. $\dfrac{60x^3-17x^2-4x+1}{5x^2+9x-2}$.

171. To reduce two or more fractions to their Lowest Common Denominator (L. C. D.) we have the following rule;

Find the L. C. M. of all the denominators for the L. C. D. Divide the L. C. D. by the denominator of the first fraction and multiply its numerator by the quotient for a new numerator. Deal in a similar manner with each of the other fractions.

EXAMPLE 1. Reduce $\dfrac{x}{a^2b(x+a)}$, $\dfrac{y}{ab^2(x-a)}$, and $\dfrac{z}{ab(x^2-a^2)}$ to equivalent fractions having the L. C. D.

The L. C. M. of the denominators is $a^2b^2(x^2-a^2)$.

Dividing this L. C. D. by the denominator of each fraction, and multiplying both the numerator and denominator by the quotient, we obtain

$$\frac{x}{a^2b(x+a)}=\frac{x\times b(x-a)}{a^2b^2(x^2-a^2)}=\frac{bx(x-a)}{a^2b^2(x^2-a^2)},$$

$$\frac{y}{ab^2(x-a)}=\frac{y\times a(x+a)}{a^2b^2(x^2-a^2)}=\frac{ay(x+a)}{a^2b^2(x^2-a^2)},$$

and $$\frac{z}{ab\,(x^2-a^2)}=\frac{z\times ab}{a^2b^2\,(x^2-a^2)}=\frac{abz}{a^2b^2\,(x^2-a^2)}.$$

EXAMPLE 2. Reduce $\frac{1}{x^2-5x+6}, \frac{1}{x^2-4x+3}, \frac{1}{x^2-3x+2}$ to equivalent fractions having the L. C. D.

The denominators equal $(x-3)(x-2)$, $(x-3)(x-1)$, $(x-2)(x-1)$, respectively. Hence the L. C. D. is $(x-3)(x-2)(x-1)$.

$$\therefore \frac{1}{(x-3)(x-2)}=\frac{x-1}{(x-3)(x-2)(x-1)};$$

$$\frac{1}{(x-3)(x-1)}=\frac{x-2}{(x-3)(x-2)(x-1)};$$

$$\frac{1}{(x-2)(x-1)}=\frac{x-3}{(x-3)(x-2)(x-1)}.$$

Exercise 54.

Reduce to equivalent fractions having the L. C. D.:

1. $\frac{3}{4x}, \frac{4}{6x^2}, \frac{5}{12x^3}$.
2. $\frac{5a^2}{6x^2y}, \frac{3bx}{8y^2x}, \frac{7cy-m}{10xz^2}$.
3. $\frac{a}{x-a}, \frac{x}{x-a}, \frac{a^2}{x^2-a^2}$.
4. $\frac{2}{a-b}, \frac{3}{a+b}, \frac{4}{a^2+b^2}$.
5. $\frac{ay}{1-x}, \frac{ax^2}{(1-x)^2}, \frac{xy^3}{(1-x)^3}$.
6. $\frac{m}{n}, \frac{m^3}{mn-n^2}, \frac{mn^2}{m^2-n^2}$.
7. $\frac{ab}{am-bm+an-bn}, \frac{m-n}{2a^2-2ab}$.
8. $\frac{3}{x^2+3x+2}, \frac{5}{x^2+2x-3}, \frac{2}{x^2+5x+6}$.
9. $\frac{2a}{a-b}, \frac{b}{2b-2a}, \frac{3a^2}{4(a^2-b^2)}, \frac{5b^2}{6(b^2-a^2)}$.
10. $\frac{1}{(x-a)(x-b)}, \frac{1}{(b-x)(c-x)}, \frac{1}{(x-c)(x-a)}$.

172. Addition and Subtraction of Fractions. By the distributive law of division and by § 83 we have

$$\frac{a}{x}+\frac{b}{x}-\frac{c}{x}=\frac{a+b-c}{x},$$

and
$$a+\frac{b}{y}-\frac{c}{z}=\frac{ayz}{yz}+\frac{bz}{yz}-\frac{cy}{yz}$$
$$=\frac{ayz+bz-cy}{yz}.$$

Hence, for adding or subtracting fractions, we have the following rule:

Reduce the fractions, if necessary, to equivalent fractions having the L. C. D.; then add or subtract each numerator as the sign before the fraction directs, and write the result over the L. C. D.

NOTE. The student should remember that when either the numerator or the denominator is a polynomial, the dividing line in a fraction is a sign of grouping as well as a sign of division.

EXAMPLE 1. Find the value of $\frac{1}{x-y}+\frac{1}{x+y}$.

The L. C. M. of the denominators is $(x-y)(x+y)$; and

$$\frac{1}{x-y}+\frac{1}{x+y}=\frac{x+y}{(x-y)(x+y)}+\frac{x-y}{(x-y)(x+y)}$$
$$=\frac{x+y+(x-y)}{x^2-y^2}=\frac{2x}{x^2-y^2}.$$

EXAMPLE 2. Find the value of $\frac{a^2-bc}{bc}-\frac{ac-b^2}{ca}-\frac{ab-c^2}{ab}$.

The L. C. M. of the denominators is abc, hence

$$\text{the expression }=\frac{a(a^2-bc)}{abc}-\frac{b(ac-b^2)}{abc}-\frac{c(ab-c^2)}{abc}$$
$$=\frac{a(a^2-bc)-b(ac-b^2)-c(ab-c^2)}{abc}$$
$$=\frac{a^3+b^3+c^3-3abc}{abc}.$$

Exercise 55.

Find the value of

1. $\frac{a-5b}{5}-\frac{a-3b}{3}$.

2. $\frac{a-3b}{4}+\frac{3a-b}{5}$.

3. $\frac{6a-5b}{3}-\frac{4a-7b}{2}$.

4. $\frac{a-b}{a^2b}+\frac{a-b}{ab^2}$.

5. $\frac{x}{4}-\frac{x-4}{3}+\frac{x-5}{6}$.

6. $2\frac{x-y}{3}-3\frac{x+y}{4}$.

7. $5\frac{x-2y}{6}-3\frac{y-2x}{4}$.

8. $5\frac{a-b}{c}-7\frac{a-b}{x}$.

9. $\frac{2x-3}{9}-\frac{x+2}{6}+\frac{5x+8}{12}$.

10. $\frac{a-2b}{2a}-\frac{a-5b}{4a}+\frac{a+7b}{8a}$.

11. $\frac{b+c}{2a}+\frac{c+a}{4b}-\frac{a-b}{3c}$.

12. $\frac{a-x}{x}+\frac{a+x}{a}-\frac{a^2-x^2}{2ax}$.

13. $\frac{2a^2-b^2}{a^2}-\frac{b^2-c^2}{b^2}-\frac{c^2-a^2}{c^2}$.

14. $\frac{2x-3y}{xy}+\frac{3x-2z}{xz}+\frac{5}{x}$.

15. $\frac{x-3}{5x}+\frac{x^2-9}{10x^2}-\frac{8-x^3}{15x^3}$.

16. $\frac{2}{xy}-\frac{3y^2-x^2}{xy^3}+\frac{xy+y^2}{x^2y^2}$.

17. Reduce $a+b-\frac{a^2+b^2}{a-b}$ to a fractional form.

$$\frac{a+b}{1}-\frac{a^2+b^2}{a-b}=\frac{a^2-b^2-(a^2+b^2)}{a-b}=\frac{-2b^2}{a-b}=\frac{2b^2}{b-a}.$$

Reduce to a fractional form:

18. $a - 1 + \dfrac{2}{a + 1}.$

19. $a + x + \dfrac{x^2}{a - x}.$

20. $x + 2y + \dfrac{4y^2}{x - 2y}.$

21. $x^2 - 3x - \dfrac{3x(3 - x).}{x - 2}$

22. $a^2 - 2ax + 4x^2 - \dfrac{6x^3}{a + 2x}.$

23. $x - a + y + \dfrac{a^2 - ay + y^2}{x + a}.$

24. $1 + x + x^2 + x^3 + \dfrac{x^4}{1 - x}.$

25. $\dfrac{x^4 + 2x^2 + 1}{x^2 + x + 1} - (x - x^2 - 1).$

26. $1 + 2x + 4x^2 + \dfrac{x^3 + 1}{2x - 1}.$

27. $x^2 - 2x + 3 - \dfrac{x^3 + 13x - 5}{x^2 + 3x - 2}.$

28. Find the value of $\dfrac{a}{a - x} + \dfrac{ax}{x^2 - a^2}.$

$$\frac{a}{a - x} + \frac{ax}{x^2 - a^2} = \frac{a}{a - x} + \frac{-ax}{a^2 - x^2} \qquad \S\,88$$

$$= \frac{a(a + x) - ax}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.$$

NOTE. Beginners should always arrange the denominators of the fractions to be added or subtracted according to the descending or the ascending powers of some particular letter.

Simplify

29. $\dfrac{a}{a - b} + \dfrac{b}{b - a}.$

30. $\dfrac{x}{x - a} + \dfrac{a}{a - x}.$

31. $\dfrac{x}{x^2 - a^2} + \dfrac{a}{a^2 - x^2}.$

32. $\dfrac{1}{1 - x} - \dfrac{2}{1 - x^2}.$

33. $\dfrac{1}{3+x}+\dfrac{6}{x^2-9}$.

35. $\dfrac{1}{x-2}+\dfrac{2}{(x-2)^2}$.

34. $\dfrac{1+x}{1-x}-\dfrac{1-x}{1+x}$.

36. $\dfrac{a+2b}{a-2b}-\dfrac{a-2b}{a+2b}$.

37. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}+\dfrac{4y^2}{x^2-y^2}$.

38. $\dfrac{2a}{a+b}+\dfrac{2b}{a-b}-\dfrac{a^2+b^2}{a^2-b^2}$.

39. $\dfrac{a}{a-1}-1-\dfrac{1}{a(a-1)}$.

40. $\dfrac{4a^2+b^2}{4a^2-b^2}-\dfrac{2a-b}{2a+b}$.

41. $\dfrac{1}{x-1}+\dfrac{1}{x-2}-\dfrac{1}{x+2}-\dfrac{1}{x+1}$.

Here the work is simplified by first combining the first and the fourth fraction, next the second and the third, and then adding these results as below:

$$\frac{1}{x-1}-\frac{1}{x+1}=\frac{x+1-(x-1)}{x^2-1}=\frac{2}{x^2-1},$$

$$\frac{1}{x-2}-\frac{1}{x+2}=\frac{x+2-(x-2)}{x^2-4}=\frac{4}{x^2-4},$$

and $$\frac{2}{x^2-1}+\frac{4}{x^2-4}=\frac{2(x^2-4)+4(x^2-1)}{(x^2-1)(x^2-4)}=\frac{6x^2-12}{x^4-5x^2+4}.$$

42. $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1+x^4}$.

Here the work is simplified by proceeding as below:

$$\frac{1}{1-x}+\frac{1}{1+x}=\frac{1+x+(1-x)}{1-x^2}=\frac{2}{1-x^2};$$

$$\frac{2}{1-x^2}+\frac{2}{1+x^2}=\frac{2(1+x^2)+2(1-x^2)}{1-x^4}=\frac{4}{1-x^4};$$

$$\frac{4}{1-x^4}+\frac{4}{1+x^4}=\frac{4(1+x^4)+4(1-x^4)}{1-x^8}=\frac{8}{1-x^8}.$$

43. $\dfrac{2}{x-2}-\dfrac{1}{x+2}-\dfrac{x+6}{x^2+4}.$

44. $\dfrac{a}{a-x}+\dfrac{a}{a+x}+\dfrac{2a^2}{a^2+x^2}.$

45. $\dfrac{3-x}{1-3x}-\dfrac{3+x}{1+3x}-\dfrac{1-16x}{9x^2-1}.$

46. $\dfrac{1}{x-1}-\dfrac{1}{2(x+1)}-\dfrac{x+3}{2(x^2+1)}.$

47. $\dfrac{a}{a-x}+\dfrac{a}{a+x}+\dfrac{2a^2}{a^2+x^2}+\dfrac{4a^4}{a^4+x^4}.$

48. $\dfrac{1}{x-3}-\dfrac{3}{x-1}+\dfrac{3}{x+1}-\dfrac{1}{x+3}.$

49. $\dfrac{1}{x-2}-\dfrac{4}{x-1}+\dfrac{6}{x}-\dfrac{4}{x+1}+\dfrac{1}{x+2}.$

50. $\dfrac{2}{x^2-3x+2}+\dfrac{2}{x^2-x-2}-\dfrac{1}{x^2-1}.$

The expression

$$=\frac{2}{(x-2)(x-1)}+\frac{2}{(x-2)(x+1)}-\frac{1}{(x-1)(x+1)}$$

$$=\frac{2(x+1)+2(x-1)-(x-2)}{(x-2)(x-1)(x+1)}$$

$$=\frac{3x+2}{(x-2)(x-1)(x+1)}.$$

51. $\dfrac{1}{x^2-9x+20}+\dfrac{1}{x^2-11x+30}.$

52. $\dfrac{1}{x^2-7x+12}-\dfrac{1}{x^2-5x+6}.$

53. $\dfrac{1}{2x^2-x-1}-\dfrac{1}{2x^2+x-3}.$

54. $\frac{1}{2x^2-x-1}-\frac{3}{6x^2-x-2}.$

55. $\frac{4}{4-7a-2a^2}-\frac{3}{3-a-10a^2}.$

56. $\frac{5}{5+x-18x^2}-\frac{2}{2+5x+2x^2}.$

57. $\frac{5x}{2(x+1)(x-3)}-\frac{15(x-1)}{16(x-3)(x-2)}$
$-\frac{9(x+3)}{16(x+1)(x-2)}.$

58. $\frac{x}{x^2+5x+6}+\frac{15}{x^2+9x+14}-\frac{12}{x^2+10x+21}.$

59. $\frac{3}{x^2-1}+\frac{4}{2x+1}+\frac{4x+2}{2x^2+3x+1}.$

60. $\frac{24x}{9-12x+4x^2}-\frac{3+2x}{3-2x}+\frac{3-2x}{3+2x}.$

61. $\frac{1}{x^2+5ax+6a^2}-\frac{2}{x^2+4ax+3a^2}+\frac{1}{x^2+3ax+2a^2}.$

62. $\frac{2a}{(x-2a)^2}-\frac{x-a}{x^2-5ax+6a^2}+\frac{2}{x-3a}.$

63. $\frac{1}{a}-\frac{4}{a+1}+\frac{6}{a+2}-\frac{4}{a+3}+\frac{1}{a+4}.$

64. $\frac{1}{x^2-5x+6}-\frac{2}{x^2-4x+3}+\frac{1}{x^2-3x+2}.$

65. $\frac{1}{8-8x}-\frac{1}{8+8x}+\frac{x}{4+4x^2}-\frac{x}{2+2x^4}.$

66. $\frac{1}{6a-18}-\frac{1}{6a+18}-\frac{1}{a^2+9}+\frac{18}{a^4+81}.$

67. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}.$

Change the quality of the factor $b-a$ in the denominator of the second fraction and also the sign before the fraction. Also change the quality of both the factors in the denominator of the third fraction. The expression then becomes

$$\frac{1}{(a-b)(a-c)}-\frac{1}{(b-c)(a-b)}+\frac{1}{(a-c)(b-c)},$$

in which a precedes b or c, and b precedes c in all the factors. Now the L. C. M. of the denominators is $(a-b)(a-c)(b-c)$;

$$\therefore \text{ the expression} = \frac{b-c-(a-c)+(a-b)}{(a-b)(a-c)(b-c)}=0.$$

68. $\dfrac{c}{(b-c)(b-a)}+\dfrac{a}{(c-a)(c-b)}+\dfrac{b}{(a-b)(a-c)}.$

69. $\dfrac{z}{(y-x)(z-x)}+\dfrac{x}{(y-z)(y-x)}+\dfrac{y}{(z-x)(z-y)}.$

70. $\dfrac{y+z}{(y-x)(z-x)}+\dfrac{z+x}{(y-z)(y-x)}+\dfrac{x+y}{(z-x)(z-y)}.$

173. Multiplication and Division of Fractions.

EXAMPLE 1. Simplify $\dfrac{3a^2b}{4c^2d}\times\dfrac{14cd^3}{9a^2b^2}\div\dfrac{7b^2c^2}{6acd^2}.$

The expression $=\dfrac{3a^2b}{4c^2d}\times\dfrac{14cd^3}{9a^2b^2}\times\dfrac{6acd^2}{7b^2c^2}$ § 85

$$=\frac{\cancel{3}\cdot\cancel{14}\cdot\cancel{6}\cdot\overset{a}{\cancel{a^3}}\cdot b\cdot c^2\cdot\overset{d^4}{\cancel{d^5}}}{\cancel{4}\cdot\cancel{9}\cdot\cancel{7}\cdot a^2\cdot\underset{b^3\,c^2}{\cancel{b^4}\cancel{c^4}}d}=\frac{ad^4}{b^3c^2}$$ §§ 75, 83

The common factors may be cancelled before the multiplication is performed as in the next example.

Example 2. Simplify $\frac{x^2-xy}{3y^2} \div \frac{x(x+2y)}{xy+y^2} \times \frac{2xy}{x^2-y^2}$.

The expression $= \frac{\cancel{x(x-y)}}{3\cancel{y^2}} \times \frac{\cancel{y(x+y)}}{\cancel{x}(x+2y)} \times \frac{2x\cancel{y}}{\cancel{(x-y)}\cancel{(x+y)}}$

$= \frac{2x}{3(x+2y)}$.

Exercise 56.

Simplify

1. $\frac{x-y}{x^2+xy} \times \frac{x+y}{xy-y^2}$.

2. $\frac{x^2+2x}{x^2-9} \times \frac{x^2-3x}{x^2-4}$.

3. $\frac{x^2-y^2}{x^2-4y^2} \times \frac{x-2y}{x+y}$.

4. $\frac{a+b}{a^3-a^2b} \times \frac{ab-b^2}{ab+a^2}$.

5. $\frac{x^3+3x^2}{x+4} \div \frac{x+3}{x^2+4x}$.

6. $\frac{a+4b}{a^2+5ab} \div \frac{ab+4b^2}{a^3+5a^2b}$.

7. $\frac{x-1}{x-2} \times \frac{x-2}{x-3} \div \frac{x-4}{x-3}$.

8. $\frac{x^3-a^3}{x^2-4a^2} \times \frac{x+2a}{x-a}$.

9. $\frac{a^3-x^3}{a^3+x^3} \div \frac{(a-x)^2}{a^2-x^2}$.

10. $\frac{14x^2-7x}{12x^3+24x^2} \div \frac{2x-1}{x^2+2x}$.

11. $\frac{16x^2-9a^2}{x^2-4} \div \frac{4x-3a}{x-2}$.

12. $\frac{a^2b^2+3ab}{4a^2-1} \div \frac{ab+3}{2a+1}$.

13. $\frac{x^2-14x-15}{x^2-4x-45} \div \frac{x^2-12x-45}{x^2-6x-27}$.

14. $\frac{x^3-6x^2+36x}{x^2-49} \div \frac{x^4+216x}{x^2-x-42}$.

15. $\frac{x^2-x-20}{x^2-25} \div \frac{x+1}{x^2+5x} \div \frac{x^2+2x-8}{x^2-x-2}$.

16. $\frac{x^2-18x+80}{x^2-5x-50} \div \frac{x^2-15x+56}{x^2-6x-7} \times \frac{x+5}{x-1}$.

17. $\frac{x^2-8x-9}{x^2-17x+72} \times \frac{x^2-25}{x^2-1} \div \frac{x^2+4x-5}{x^2-9x+8}$.

18. $\dfrac{x^4-8x}{x^2-4x-5}\times\dfrac{x^2+2x+1}{x^3-x^2-2x}\div\dfrac{x^2+2x+4}{x-5}.$

19. $\dfrac{(a+b)^2}{(a-b)^3}\div\dfrac{a^4-b^4}{(a^2-b^2)^2}\div\dfrac{(a+b)^3}{a^2+b^2}.$

20. $\dfrac{(a-b)^2-c^2}{(a-c)^2-b^2}\div\dfrac{c^2-(a-b)^2}{b^2-(c-a)^2}.$

21. $\dfrac{x^6+y^6}{x^6-y^6}\times\dfrac{x-y}{x+y}\div\dfrac{x^4-x^2y^2+y^4}{x^4+x^2y^2+y^4}.$

22. $\left(1-\dfrac{2xy}{x^2+y^2}\right)\div\left(\dfrac{x^3-y^3}{x-y}-3xy\right).$

First reduce each of the mixed expressions to fractions.

23. $\left(a+\dfrac{ab}{a-b}\right)\left(b-\dfrac{ab}{a+b}\right).$

24. $\dfrac{x^2+xy}{x^2+y^2}\times\left(\dfrac{x}{x-y}-\dfrac{y}{x+y}\right).$

25. $\left(\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}\right)\div\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right).$

26. $\left(a+\dfrac{3x^2}{a}\right)\times\left(\dfrac{a^2}{3x^2}-1\right)\div\dfrac{a}{x}.$

27. $\dfrac{a^3+b^3+3ab(a+b)}{2a(a-b)^2}\times b^2\left(a-\dfrac{b^2}{a}\right)\div\left(1+\dfrac{b}{a}\right)^4.$

28. $\dfrac{4x^2+x-14}{6xy-14y}\times\dfrac{4x^2}{x^2-4}\times\dfrac{x-2}{4x-7}\div\dfrac{2x^2+4x}{3x^2-x-14}.$

29. $\dfrac{x^2+x-2}{x^2-x-20}\times\dfrac{x^2+5x+4}{x^2-x}\div\left(\dfrac{x^2+3x+2}{x^2-2x-15}\times\dfrac{x+3}{x^2}\right).$

30. $\dfrac{4x^2-16x+15}{2x^2+3x+1}\times\dfrac{x^2-6x-7}{2x^2-17x+21}\times\dfrac{4x^2-1}{4x^2-20x+25}.$

31. $\dfrac{(a+b)^2-c^2}{a^2+ab-ac}\times\dfrac{a}{(a+c)^2-b^2}\times\dfrac{(a-b)^2-c^2}{ab-b^2-bc}.$

32. $\dfrac{a^2+2ab+b^2-c^2}{a^2-b^2-c^2-2bc}\times\dfrac{a^2-2ac+c^2-b^2}{b^2-2bc+c^2-a^2}.$

33. $\dfrac{x^2-64}{x^2+24x+128}\times\dfrac{x^2+12x-64}{x^3-64}\div\dfrac{x^2-16x+64}{x^2+4x+16}.$

174. Complex Fractions. The quotient of one fractional expression divided by another is often called a *Complex Fraction.* This quotient is usually indicated by writing the divisor under the dividend.

Thus $\dfrac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{x}{y}-\dfrac{d}{e}}$, or $\left(\dfrac{a}{b}+\dfrac{c}{d}\right)\div\left(\dfrac{x}{y}-\dfrac{d}{e}\right)$ is a complex fraction.

Sometimes the shortest way to simplify a complex fraction is to multiply both its dividend and divisor by the L. C. M. of the denominators of their fractional terms.

Thus
$$\frac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{x}{y}-\dfrac{c}{b}}=\frac{\left(\dfrac{a}{b}+\dfrac{c}{d}\right)bdy}{\left(\dfrac{x}{y}-\dfrac{c}{b}\right)bdy}$$
$$=\frac{ady+bcy}{bdx-cdy}.$$

Here bdy is the L. C. M. of the denominators of the fractional terms in the dividend and divisor of the complex fraction.

Again,
$$\frac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}=\frac{\left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)(a-x)(a+x)}{\left(\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}\right)(a-x)(a+x)}$$
$$=\frac{(a+x)^2-(a-x)^2}{(a+x)^2+(a-x)^2}$$
$$=\frac{2ax}{a^2+x^2}.$$

Also,

$$\frac{x+\dfrac{a^2}{x}}{x-\dfrac{a^4}{x^3}}=\frac{\left(x+\dfrac{a^2}{x}\right)x^3}{\left(x-\dfrac{a^4}{x^3}\right)x^3}$$

$$=\frac{x^4+a^2x^2}{x^4-a^4}=\frac{x^2}{x^2-a^2}.$$

Example. Simplify $\dfrac{x}{x-\dfrac{x+2}{x+2-\dfrac{x+1}{x}}}$.

In a fraction of this kind, called a *Continued Fraction*, we begin with the lowest complex fraction and simplify step by step. Thus

$$\frac{x}{x-\dfrac{x+2}{x+2-\dfrac{x+1}{x}}}=\frac{x}{x-\dfrac{(x+2)\,x}{(x+2)\,x-(x+1)}} \qquad \S\ 83$$

$$=\frac{x}{x-\dfrac{x^2+2x}{x^2+x-1}}$$

$$=\frac{x\,(x^2+x-1)}{x(x^2+x-1)-(x^2+2x)}$$

$$=\frac{x^2+x-1}{x^2-3}.$$

Exercise 57.

Find the value of

1. $\dfrac{\dfrac{m}{n}-\dfrac{l}{m}}{\dfrac{a}{m}-\dfrac{b}{n}}.$

2. $\dfrac{a+\dfrac{b}{d}}{x-\dfrac{y}{d}}.$

3. $\dfrac{2+\dfrac{3a}{4b}}{a+\dfrac{8b}{3}}.$

4. $\dfrac{3a+\dfrac{7b}{8c}}{3c+\dfrac{7b}{8a}}.$

5. $\dfrac{x}{x-\dfrac{m}{n}}.$

6. $\dfrac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}}.$

7. $\dfrac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{m}{n}+\dfrac{k}{p}}.$

8. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}.$

9. $\dfrac{x+5+\dfrac{6}{x}}{1+\dfrac{6}{x}+\dfrac{8}{x^2}}.$

10. $\dfrac{\dfrac{1}{x}-\dfrac{2}{x^2}-\dfrac{3}{x^3}}{\dfrac{9}{x}-x}.$

11. $\dfrac{2x^2-x-6}{\dfrac{4}{x^2}-1}.$

12. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{1-\dfrac{a^2+b^2}{(a+b)^2}}.$

13. $\dfrac{\dfrac{a}{x^2}+\dfrac{x}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ax}+\dfrac{1}{x^2}}.$

14. $1+\dfrac{x}{1+x\dfrac{2x^2}{1-x}}.$

15. $\dfrac{1}{a-\dfrac{a^2-1}{a+\dfrac{1}{a-1}}}.$

16. $\dfrac{a}{x+\dfrac{m}{y+\dfrac{n}{z}}}.$

17. $\dfrac{1}{1-\dfrac{1+x}{x-\dfrac{1}{x}}}.$

18. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}.$

19. $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x}}}-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}.$

20. $\left\{x-y-\dfrac{1}{x-y+\dfrac{xy}{x-y}}\right\}\times\dfrac{x^3+y^3}{x^2-y^2}.$

21. $\left\{x+y-\dfrac{1}{x+y-\dfrac{xy}{x+y}}\right\}\times\dfrac{x^3-y^3}{x^2-y^2}.$

22. $\left\{1-\dfrac{1-x}{1+x}+\dfrac{1+2x^2}{1-x^2}\right\}\times\dfrac{x+1}{2x+1}.$

23. $$\frac{x-2-\dfrac{1}{x-2}}{x-2-\dfrac{4}{x-5}}\times\frac{x-4-\dfrac{4}{x-4}}{x-4-\dfrac{1}{x-4}}.$$

24. $$\left(\frac{a^2+b^2}{2ab}-1\right)\frac{ab^2}{a^3+b^3}\div\frac{4ab(a+b)}{a^2-ab+b^2}.$$

25. $$\frac{\left\{1+\dfrac{c}{a+b}+\dfrac{c^2}{(a+b)^2}\right\}\left\{1-\dfrac{c^2}{(a+b)^2}\right\}}{\left\{1-\dfrac{c^3}{(a+b)^3}\right\}\left\{1+\dfrac{c}{a+b}\right\}}.$$

26. $$\left(\frac{x^2}{y}-\frac{y^2}{x}\right)\left(\frac{x}{y^2}+\frac{y}{x^2}\right)\div\left(\frac{x^2}{y}+\frac{y^2}{x}\right)\left(\frac{x}{y^2}-\frac{y}{x^2}\right).$$

27. $$\frac{a^6+b^6}{a^6-b^6}\times\frac{a-b}{a^2-b^2}\div\frac{a^4-a^2b^2+b^4}{a^4+a^2b^2+b^4}.$$

28. $$\left(\frac{x+2}{2x+3}-\frac{4x+5}{5x+6}\right)\div\left(\frac{2x+3}{3x+4}-\frac{3x+4}{4x+5}\right).$$

29. $$\left(\frac{1+x}{1+x^2}-\frac{1+x^2}{1+x^3}\right)\div\left(\frac{1+x^2}{1+x^3}-\frac{1+x^3}{1+x^4}\right).$$

30. $$\left\{\left(\frac{a-b}{a+b}\right)^2-2\frac{a-b}{a+b}+1\right\}\div\left\{\left(\frac{a+b}{a-b}\right)^2-2\frac{a+b}{a-b}+1\right\}.$$

CHAPTER XIV.

QUADRATIC EQUATIONS.

175.* *A product is zero when one of its factors is zero; and a product cannot be zero unless one, or more, of its factors is zero.*

Thus xy is zero, if x is zero, or if y is zero; and if we know that xy is zero, we conclude that either x or y is zero.

Similarly, the product $x(x-1)(x-2)$ is zero, if x is zero, if $x-1$ is zero, or if $x-2$ is zero; and in order that this product may be zero, one of its factors must be zero.

Thus the equation,

$$(x-1)(x-3)(x+5)=0, \qquad (1)$$

is satisfied if $x-1=0$, if $x-3=0$, or if $x+5=0$, and in no other case; hence the solutions of equation (1) are 1, 3, and -5.

EXAMPLE 1. Solve $x^2=4x+12. \qquad (1)$

Transpose, $x^2-4x-12=0. \qquad (2)$

Factor, $x^2-4x-12\equiv(x+2)(x-6).$

Substitute in (2), $(x+2)(x-6)=0. \qquad (3)$

Now (3) is satisfied if $x+2=0$ or if $x-6=0$; hence the solutions of (3), or of its equivalent equation (1), are -2 and 6.

EXAMPLE 2. Solve $x^3+x^2=6x. \qquad (1)$

Transpose, $x^3+x^2-6x=0. \qquad (2)$

Factor, $x(x-2)(x+3)=0. \qquad (3)$

Now (3) is equivalent to the three equations

$$x=0,\ x-2=0, \text{ and } x+3=0;$$

hence the solutions of (3), or (1), are 0, 2, and -3.

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* This theorem assumes that the factors are all finite.

Example 3. Solve $3x^2 + 19x + 20 = 0.$ (1)

Multiply by 3, $(3x)^2 + 19(3x) + 60 = 0.$ (2)

Factor, $(3x + 4)(3x + 15) = 0.$ (3)

Now (3) is equivalent to the two equations

$$3x + 4 = 0 \text{ and } 3x + 15 = 0;$$

hence the solutions of (3), or (1), are $-\frac{4}{3}$ and -5.

Example 4. Solve $4x^4 - 37x^2 + 9 = 0.$ (1)

Multiply by 4, $(4x^2)^2 - 37(4x^2) + 36 = 0.$ (2)

Factor, $(4x^2 - 36)(4x^2 - 1) = 0;$

or, $(2x - 6)(2x + 6)(2x - 1)(2x + 1) = 0.$ (3)

Hence the solutions of (3), or (1), are 3, -3, $\frac{1}{2}$, and $-\frac{1}{2}$.

176. From the last article it follows that to solve an equation in x, we may proceed as follows:

First transpose all its terms to the first member. Then resolve this member into its integral factors in x.

The values of x *obtained by putting each of these factors equal to zero will be the required solutions.*

177. An equation can be formed of which the solutions shall be any given numbers, by subtracting each of the numbers from x, and putting the product of these results equal to zero.

Thus the equation whose solutions are 2, 3, and -1 is

$$(x - 2)(x - 3)(x + 1) = 0,$$

or $$x^3 - 4x^2 + x + 6 = 0.$$

Note. What precedes suggests the truth that an equation of the third degree has 3 solutions; one of the fourth degree has 4 solutions; and so on.

Exercise 58.

Solve each of the following equations:

1. $x^2 - 7x = 0.$

2. $x^2 + 9x = 0.$

3. $x^2 = 4x + 12.$

4. $x^2 + 12x = -35.$

5. $x^2 = 6x + 91.$

6. $x^2 + 12 = 7x.$

7. $x^2 + 20 = 12x.$

8. $x^2 + 20 = 9x.$

9. $x^2 + 28 = 11x.$

10. $x^2 + 150 = 25x.$

11. $3x^2 = 10x - 3.$

12. $3x^2 + 11x = 20.$

13. $4x^2 + 21x = 18.$

14. $3x^2 - 2x = 96.$

15. $15x^2 + 4x = 3.$

16. $6x^2 - 7x = 3.$

17. $19x = 4 - 5x^2.$

18. $5x^2 - 4x = 33.$

19. $x^2 + ax = 42a^2.$

20. $x^2 - 20ax = 96a^2.$

21. $8x^2 + x = 30.$

22. $x + 22 = 6x^2.$

23. $21 + x = 2x^2.$

24. $3x^2 + 35 = 22x.$

25. $6x^2 + 55x = 50.$

26. $6x^2 + 6 = 13x.$

27. $19x^2 - 39x = -2.$

28. $15x^2 - 2ax = a^2.$

29. $17x^2 + 8 = 70x.$

30. $21x^2 + 10x = -1.$

31. $6x^2 = 11kx + 7k^2.$

32. $x^2 - 23ax = -132a^2.$

33. $2x^2 - 3a^2 = 5ax.$

34. $12x^2 + 3a^2 = 13ax.$

35. $132x^2 + x = 1.$

36. $x^2 + 600a^2 = -49ax.$

37. $x^3 - 3x^2 = 10x.$

38. $16x^3 + 3x = 16x^2.$

39. $110x^3 + x = 21x^2.$

40. $5x^3 = 8x^2 + 21x.$

41. $32x - 3x^3 = 10x^2.$

42. $x^3 + 2a^2x = 3ax^2.$

43. $x^3 - x^2 + 9 = 9x.$

44. $x^3 + 2x^2 - 16x = 32.$

45. $x^4 - 26x^2 + 25 = 0.$

46. $x^4 + 36 = 13x^2.$

47. $36x^4 + 1 = 13x^2.$

48. $x^4 - a^2x^2 + 4a^2c^2 = 4c^2x^2.$

Form the equation whose solutions are

49. $1, -3.$

50. $2, -7.$

51. $-9, -11.$

52. $-2, -13.$

53. $-1, -2, -3.$

54. $2, -4, -5.$

55. $5, -3, -7.$

56. $3, -2, -5.$

178. Completing the Square. Since

$$x^2 + 2ax + a^2 = (x + a)^2,$$

it follows that, to obtain a perfect square, we must add to the binomial $x^2 + 2ax$ the term a^2, that is, *the square of half the coefficient of* x.

Thus to obtain a perfect square, we must add to the binomial $x^2 - 6x$ the term $(6 \div 2)^2$, or 9; that is,

$$x^2 - 6x + 9 = (x - 3)^2.$$

Again, to obtain a perfect square, we must add to the binomial $a^2x^2 + 8abx$, or $(ax)^2 + 8b(ax)$, the term $(8b \div 2)^2$, or $16b^2$; that is,

$$a^2x^2 + 8abx + 16b^2 = (ax + 4b)^2.$$

179. *If the square root of both members of an equation be extracted, and the double sign ± be written before one member, the two resulting equations will be equivalent to the given equation; that is, the equation*

$$A^2 = B^2 \qquad (1)$$

is equivalent to the two equations

$$A = \pm B. \qquad (2)$$

For if in equation (1) we transpose B^2 and factor, we obtain the equivalent equation,

$$(A - B)(A + B) = 0. \qquad (3)$$

Now (3) is equivalent to the two equations in (2).

180. The principles of §§ 178 and 179 are very useful in solving quadratic equations that cannot be solved by inspection, as in § 175.

EXAMPLE 1. Solve $x^2 + 32 - 10x = 0.$ (1)

Transpose 32, $x^2 - 10x = -32.$ (2)

Add $(10 \div 2)^2$, $x^2 - 10x + 25 = 25 - 32 = -7.$ (3)

Extract square root, $x - 5 = \pm\sqrt{-7}.$ (4)

$\therefore x = 5 \pm \sqrt{-7}.$ (5)

Now by § 179 equations (3) and (4) are equivalent; and, by § 100, (1), (2), and (3) are equivalent, as also (4) and (5); hence the solutions of (1) are $5 + \sqrt{-7}$ and $5 - \sqrt{-7}$.

EXAMPLE 2. Solve $\frac{x^2 - 9}{4} = \frac{x^2 + 1}{5}.$ (1)

Multiply by 20, $5x^2 - 45 = 4x^2 + 4.$

Transpose, $x^2 = 49.$

Extract square root, $x = \pm 7.$

Hence the solutions of (1) are $+7$ and -7.

EXAMPLE 3. Solve $2x^2 = 6x + 10.$ (1)

Transpose $6x$, $2x^2 - 6x = 10.$

Divide by 2, $x^2 - 3x = 5.$

Complete the square, $x^2 - 3x + \frac{9}{4} = 5 + \frac{9}{4} = \frac{1}{4}\cdot 29.$

Extract the square root, $x - \frac{3}{2} = \pm \frac{1}{2}\sqrt{29}.$

$\therefore x = \frac{1}{2}(3 \pm \sqrt{29}).$ (2)

Both solutions involve the surd $\sqrt{29}$. By extracting the square root of 29, we may find their value correct to any required number of places of decimals.

Thus $\sqrt{29} = 5.3851$ approximately.

Hence $x = \frac{1}{2}(3 \pm 5.3851) = 4.1925$ or -1.1925.

These values of x are correct to four places of decimals. Unless, however, the numerical values are required, it is usual to leave solutions in the form given in (2).

Hence, to solve any quadratic equation, we may proceed as follows:

First put the equation in the form of $x^2 + ax = c$.

Next add to each member the square of half the coefficient of x.

Then extract the square root of each member, writing the double sign, ±, *before the second member.*

Finally solve the two resulting linear equations.

Exercise 59.

Solve each of the following equations:

1. $x^2 + 1 = 4x$.
2. $x^2 - 2x = 4$.
3. $x^2 + 5 = 8x$.
4. $x^2 + 2x = 2$.
5. $x^2 + 6x = -3$.
6. $4x^2 + 4x = 11$.
7. $9x^2 + 6x = 17$.
8. $4x^2 - 4x - 7 = 0$.
9. $x^2 - 6x + 2 = 0$.
10. $x^2 - 6x + 11 = 0$.
11. $x^2 + 4x + 7 = 0$.
12. $x^2 - 6x + 10 = 0$.
13. $x^2 + 8x + 21 = 0$.
14. $2x^2 - 5x + 15 = 0$.
15. $x^2 - 3ax + 2a^2 = 0$.
16. $x^2 + 2ax = b^2 + 2ab$.
17. $x^2 + 2ab = b^2 + 2ax$.
18. $4x^2 + 4ax = b^2 - a^2$.
19. $x^2 + 3a^2 = 4ax$.
20. $(x-7)^2 = 49(x+2)^2$.

First extract the square root of both members, writing the double sign ± before the second member.

181. Complete and Pure Quadratic Equations. A quadratic equation in x may have terms in x^2, terms in x, and terms not involving x, and only such terms. An equation that has all of these terms is called a *Complete Quadratic Equation;* one that has no term in the first power of x is called a *Pure Quadratic Equation.*

Thus $2x^2 - 8 = 4x$ is a *Complete* quadratic equation, while $x^2 + 8 = 0$ is a *Pure* quadratic equation.

182. *Any quadratic equation has two, and only two, solutions.*

Any quadratic equation can evidently be reduced to an equivalent equation of the form

$$x^2 + px + q = 0. \qquad (1)$$

Transposing q and adding $(\frac{1}{2}p)^2$, we obtain

$$x^2 + px + (\tfrac{1}{2}p)^2 = (\tfrac{1}{2}p)^2 - q. \qquad (2)$$

$$\therefore x + \tfrac{1}{2}p = \pm\sqrt{(\tfrac{1}{2}p)^2 - q}, \qquad (3)$$

$$\text{or } x = -\tfrac{1}{2}p \pm \sqrt{(\tfrac{1}{2}p)^2 - q}. \qquad (4)$$

By § 179 equations (2) and (3) are equivalent; hence the two, and only two, solutions of (1) are given in (4).

Equation (1) is called the *Typical* form of a quadratic equation.

Thus the equation $3x^2 + 2x = 6x + x^2 + 8$ reduced to the *typical* form becomes $x^2 - 2x - 4 = 0$.

Here $p = -2$, and $q = -4$.

183. Discussion of Solutions in (4) of § 182.

(i.) *If* $(\frac{1}{2}p)^2 - q$ *is a perfect square, both solutions will be rational and unequal.*

(ii.) *If* $(\frac{1}{2}p)^2 - q > 0$, *the two solutions will be real and unequal.*

(iii.) *If* $(\frac{1}{2}p)^2 - q = 0$, *the two solutions will be equal.*

(iv.) *If* $(\frac{1}{2}p)^2 - q < 0$, *both solutions will be imaginary.*

(v.) *If* $p = 0$, *the equation will be a Pure quadratic, and its two solutions will be numerically equal but opposite in quality.*

(vi.) *If* q = 0, *one solution will be zero and the other* − p.

(vii.) *Both solutions will be real or both imaginary.*

NOTE. The proof of these propositions is left as an exercise for the student.

184. Sum and Product of Solutions. If a quadratic equation is in the typical form,

The sum of its solutions will be equal to the coefficient of x *with its quality changed. The product of its solutions will be equal to the known term.*

Representing the two solutions of (1) in § 182 by x_1 and x_2, we have

$$x_1 = -\tfrac{1}{2}p + \sqrt{(\tfrac{1}{2}p)^2 - q}, \tag{1}$$

and

$$x_2 = -\tfrac{1}{2}p - \sqrt{(\tfrac{1}{2}p)^2 - q}. \tag{2}$$

Adding (1) and (2), to find the sum, we obtain

$$x_1 + x_2 = -p. \tag{3}$$

Multiplying (1) into (2), to find the product, we have

$$x_1 \cdot x_2 = (-\tfrac{1}{2}p)^2 - [(\tfrac{1}{2}p)^2 - q] = q. \qquad \S\ 140$$

Thus putting the equation $3x^2 = 7x + 5$ in the typical form, we have

$$x^2 - \tfrac{7}{3}x - \tfrac{5}{3} = 0.$$

Hence the sum of its solutions is $\tfrac{7}{3}$, and their product is $-\tfrac{5}{3}$.

185. Solution by a Formula. Equation (4) of § 182 affords the following simple rule for writing out the two solutions of a quadratic equation in the typical form:

The solutions equal one-half the coefficient of x, *with its quality changed, increased and diminished by the square root of the square of one-half the coefficient of* x *diminished by the known term.*

Thus to solve $x^2 + 2x - 4 = 0$, we have

$$\begin{aligned} x &= -1 \pm \sqrt{(-1)^2 - (-4)} \\ &= -1 \pm \sqrt{5}. \end{aligned}$$

By § 184 these solutions are correct if their sum is -2, and their product -4.

But $(-1+\sqrt{5}) + (-1-\sqrt{5}) = -2$,

and $$\begin{aligned} (-1+\sqrt{5})(-1-\sqrt{5}) &= (-1)^2 - (\sqrt{5})^2 \\ &= 1 - 5 = -4. \end{aligned}$$

Exercise 60.

By § 185 solve, and by § 184 verify the solutions of,

1. $x^2 - 2x = 1.$
2. $x^2 + 8x + 5 = 0.$
3. $x^2 + 4x = 1.$
4. $x^2 + 18 = 10x.$
5. $x^2 + 3 = 2x.$
6. $x^2 + 11 = 4x.$
7. $x^2 + 31 = 10x.$
8. $x^2 + 6x + 11 = 0.$
9. $x^2 + 10x + 32 = 0.$
10. $x^2 + 52 = 14x.$
11. $3x^2 + 6x = 3.$
12. $9x^2 = 6x + 26.$

186. Resolution into Factors. Any quadratic expression can be factored by writing it as the difference of two squares.

Thus $$\begin{aligned} x^2 + 4x + 9 &\equiv x^2 + 4x + 4 - (-5) \\ &\equiv (x+2)^2 - (-5) \\ &\equiv (x + 2 + \sqrt{-5})(x + 2 - \sqrt{-5}). \end{aligned}$$

Hence the solutions of the equation

$$x^2 + 4x + 9 = 0$$

are $-2 - \sqrt{-5}$ and $-2 + \sqrt{-5}$.

EXAMPLE 1. Factor $3x^2 + 2x - \frac{44}{3}$.

$$\begin{aligned} 3\left(3x^2 + 2x - \tfrac{44}{3}\right) &\equiv (3x)^2 + 2(3x) - 44 \\ &\equiv (3x+1)^2 - 45 \\ &\equiv (3x + 1 + 3\sqrt{5})(3x + 1 - 3\sqrt{5}). \end{aligned}$$

$$\therefore\ 3x + 2x - \tfrac{44}{3} \equiv (x + \tfrac{1}{3} + \sqrt{5})(3x + 1 - 3\sqrt{5}).$$

Example 2. Factor $4x^2 + 2x + 3\frac{1}{4}$.

$$\begin{aligned} 4x^2 + 2x + 3\tfrac{1}{4} &\equiv (2x)^2 + (2x) + \tfrac{1}{4} - (-3) \\ &\equiv (2x + \tfrac{1}{2})^2 - (-3) \\ &\equiv (2x + \tfrac{1}{2} + \sqrt{-3})(2x + \tfrac{1}{2} - \sqrt{-3}). \end{aligned}$$

Exercise 61.

Factor each of the following expressions, and thus find the solutions of the equation formed by putting each expression equal to zero:

1. $x^2 + 6x + 7$.
2. $x^2 + 8x + 5$.
3. $x^2 - 10x + 31$.
4. $9x^2 - 6x - 26$.
5. $3x^2 + 6x - 3$.
6. $x^2 - 14x + 52$.
7. $x^2 + 10x + 40$.
8. $x^2 - 8x + 32$.
9. $x^2 - \frac{2}{3}x - \frac{4}{3}$.
10. $x^2 + \frac{2}{5}x - 3$.
11. $x^2 - \frac{4}{3}x + 5$.
12. $3x^2 - 8x + 7$.

187. A **Literal** equation is one in which all, or some of, the known numbers are denoted by letters.

A **Numerical** equation is one in which the known numbers are all expressed by figures.

Example 1. Solve $ax^2 + bx + c = 0$.

Multiply by $4a$, $\quad (2ax)^2 + 2b(2ax) + 4ac = 0.$

By § 185, $\quad 2ax = -b \pm \sqrt{b^2 - 4ac}.$

$$\therefore x = (-b \pm \sqrt{b^2 - 4ac}) \div 2a.$$

Example 2. Solve $-3x^2 + 2ax = 3c$.

Multiply by -3, $\quad (3x)^2 - 2a(3x) + 9c = 0.$

By § 185, $\quad 3x = a \pm \sqrt{a^2 - 9c}.$

$$\therefore x = \tfrac{1}{3}(a \pm \sqrt{a^2 - 9c}).$$

EXAMPLE 3. Solve $bcx^2 + b^2x + c^2x + bc = 0$.

Multiply by $4\,bc$, $(2\,bcx)^2 + 2(b^2 + c^2)(2\,bcx) + 4\,b^2c^2 = 0$.

By § 185, $2\,bcx = -(b^2 + c^2) \pm \sqrt{(b^2 + c^2)^2 - 4\,b^2c^2}$

$= -(b^2 + c^2) \pm (b^2 - c^2)$.

$\therefore x = -c \div b$, or $-b \div c$.

Exercise 62.

Solve

1. $x^2 + 2a^2 = 3ax$.
2. $35b^2 = 9x^2 + 6bx$.
3. $x^2 - 2ax + 4ab = 2bx$.
4. $3x^2 - 2ax - bx = 0$.
5. $x^2 - 2ax + 8x = 16a$.
6. $36x^2 - 35b^2 = 12bx$.
7. $9x^2 - 6ax = a^2 - b^2$.
8. $a(x^2 + 1) = x(a^2 + 1)$.
9. $x^2 + 2(b - c)x + c^2 = 2bc$.
10. $a(x^2 - 1) + x(a^2 - 1) = 0$.
11. $x^2 - 2(a - b)x + b^2 = 2ab$.
12. $(b - c)x^2 + (c - a)x = b - a$.
13. $(a + b)x^2 + cx = a + b + c$.
14. $abx^2 - (a^2 + b^2)x + ab = 0$.
15. $(a^2 - b^2)(x^2 - 1) = 4abx$.
16. $(b^2 - a^2)(x^2 + 1) = 2(a^2 + b^2)x$.
17. $(a - x)^3 + (x - b)^3 = (a - b)^3$.
18. $(x - a + 2b)^3 - (x - 2a + b)^3 = (a + b)^3$.

CHAPTER XV.

FRACTIONAL EQUATIONS.

188. A **Fractional Equation** is one which has one or more terms that are fractional with respect to its unknown letters.

An **Integral** equation is one whose terms are all integral with respect to its unknown letters.

Thus $\frac{x^2-4}{x+7}+\frac{2x-3}{3x}=x$ is a fractional equation;

while $\frac{x^2}{2}+\frac{ax}{b}=\frac{x+2}{a+c}$ is an integral equation.

The degree of the second is known without clearing of fractions, that of the first is not.

189. *If both members of an integral equation are multiplied by the same unknown integral expression, one or more solutions will be introduced.*

Thus if we multiply both members of the equation

$$x^2+2x=-3 \tag{1}$$

by the expression $x-5$, and transpose, we obtain the equation

$$(x^2+2x+3)(x-5)=0,$$

which is equivalent to the two equations

$$x^2+2x+3=0 \text{ and } x-5=0.$$

Hence multiplying by $x-5$ introduced the solution 5; multiplying by $x-8$ would introduce the solution 8; multiplying by $x+7$ would introduce the solution -7; and so on.

Let M denote any unknown integral expression; then multiplying both members of an integral equation by M would introduce the solutions of $M=0$.

190. *If both members of a fractional equation are multiplied by the denominator of any fractional term, in general no new solution will be introduced.*

The truth of this principle is best shown in each particular equation as below.

EXAMPLE 1. Solve $\dfrac{3}{x-1}=5-x.$ (1)

Multiply by $x-1$, $3=6x-x^2-5.$

Transpose, $x^2-6x+8=0.$

$\therefore (x-4)(x-2)=0.$ (2)

Neither solution of (2) could be introduced by multiplying by $x-1$; hence the solutions of (1) are 2 and 4.

EXAMPLE 2. Solve $\dfrac{3}{x-5}+\dfrac{2x}{x-3}=5.$ (1)

Multiplying by $(x-5)(x-3)$, we obtain

$$3(x-3)+2x(x-5)=5(x-5)(x-3).$$

$$\therefore x^2-11x+28=0.$$

$$\therefore (x-4)(x-7)=0. \quad (2)$$

Neither solution of (2) could be introduced by multiplying by $x-5$ or $x-3$; hence the solutions of (1) are 4 and 7.

EXAMPLE 3. Solve $1-\dfrac{x^2}{x-1}=\dfrac{1}{1-x}-6.$ (1)

Transposing and adding the fractions, we have

$$1-\frac{x^2-1}{x-1}=-6.$$

$$\therefore 1-(x+1)=-6, \text{ or } x=6. \quad (2)$$

Hence 6 is the only solution of (1).

But if, as would be most natural to the beginner, we first multiply equation (1) by $x - 1$, we obtain

$$x - 1 - x^2 = -1 - 6x + 6.$$

Transpose, $$x^2 - 7x + 6 = 0.$$

$$\therefore (x-1)(x-6) = 0, \qquad (3)$$

of which the solutions are 1 and 6. As was shown above, to clear equation (1) of fractions it was not necessary to multiply by $x - 1$; hence multiplying by $x - 1$ introduces the solution 1.

Multiplying an equation by an unknown integral expression that is not necessary to clear the equation of fractions will introduce new solutions. Hence to avoid introducing solutions in solving fractional equations, the following suggestions should be heeded:

(i.) Fractions having a common denominator should be combined.

(ii.) Factors common to the numerator and denominator of any fraction should be cancelled.

(iii.) When multiplying by a multiple of the denominators, we should always use the L. C. M.

EXAMPLE 4. Solve $\dfrac{x}{x-2} + \dfrac{x-9}{x-7} = \dfrac{x+1}{x-1} + \dfrac{x-8}{x-6}.$ (1)

Transpose so as to have a difference in each member,

$$\frac{x}{x-2} - \frac{x+1}{x-1} = \frac{x-8}{x-6} - \frac{x-9}{x-7}.$$

Combine, $$\frac{2}{(x-2)(x-1)} = \frac{2}{(x-7)(x-6)}. \qquad (2)$$

Clear of fractions, $x^2 - 13x + 42 = x^2 - 3x + 2.$

$$\therefore 10x = 40, \text{ or } x = 4. \qquad (3)$$

Since the solution 4 could not be introduced in clearing (2) of fractions, 4 is the solution of (1).

Example 5. Solve $\frac{x-1}{x+1}+\frac{x+5}{x+7}=\frac{x+1}{x+3}+\frac{x+3}{x+5}$. (1)

Transpose, $\frac{x-1}{x+1}-\frac{x+1}{x+3}=\frac{x+3}{x+5}-\frac{x+5}{x+7}$. (2)

Combine, $\frac{1}{(x+1)(x+3)}=\frac{1}{(x+5)(x+7)}$. (3)

Clear of fractions, $x^2+12x+35=x^2+4x+3$.

$$\therefore\ 8x=-32, \text{ or } x=-4.$$

Since the solution -4 could not be introduced in clearing (3) of fractions, -4 is the solution of (1).

Or reducing the fractions in (2) to mixed expressions, we have

$$1-\frac{2}{x+1}-1+\frac{2}{x+3}=1-\frac{2}{x+5}-1+\frac{2}{x+7}.$$

$$\therefore\ -\frac{1}{x+1}+\frac{1}{x+3}=-\frac{1}{x+5}+\frac{1}{x+7}.$$

Combining these fractions, we obtain equation (3) above.

Exercise 63.

Solve

1. $\frac{5x+7}{x-1}=3x+2.$
2. $\frac{5x-1}{x+1}=\frac{3x}{2}.$
3. $\frac{3x-8}{x-2}=\frac{5x-2}{x+5}.$
4. $\frac{x+3}{2x-7}-\frac{2x-1}{x-3}=0.$
5. $\frac{5x-7}{7x-5}=\frac{x-5}{2x-13}.$
6. $\frac{4}{x-1}-\frac{5}{x+2}=\frac{3}{x}.$
7. $\frac{5}{x-2}-\frac{4}{x}=\frac{3}{x+6}.$
8. $\frac{x}{x-1}+\frac{x-1}{x}=\frac{13}{6}.$
9. $\frac{x+1}{x+2}+\frac{x+2}{x+1}=\frac{29}{10}.$
10. $\frac{7}{x+4}+\frac{1}{4-x}=\frac{5}{3}.$

11. $\frac{x}{x-2}+\frac{4}{x+1}=3.$

12. $\frac{1}{2}+\frac{1}{3+x}+\frac{1}{2+3x}=0.$

13. $\frac{5}{x+1}-\frac{10}{x+10}=\frac{2}{3x-3}.$

14. $\frac{1}{3-x}-\frac{4}{5}=\frac{1}{9-2x}.$

15. $\frac{1}{1+x}-\frac{1}{3-x}=\frac{6}{35}.$

16. $\frac{x+4}{x-4}+\frac{x-2}{x-3}=6\frac{1}{3}.$

17. $3\frac{x-1}{x+1}-2\frac{x+1}{x-1}=5.$

18. $\frac{1}{3}+\frac{1}{x+3}+\frac{1}{x+11}=0.$

19. $\frac{3}{x-1}+\frac{4}{x-3}=\frac{15}{x+3}.$

20. $\frac{2x-3}{4}=\frac{x-5}{12}+\frac{5x-16}{x-1}.$

21. $\frac{2x-2}{2x-3}+\frac{3-3x}{3x-2}=\frac{5}{8x-12}.$

22. $\frac{5x}{x-3}-\frac{6}{x+2}+\frac{19}{3-x}=0.$

23. $\frac{2x-1}{2x+1}+\frac{13}{11}=\frac{3x+5}{3x-5}.$

24. $\frac{3x}{x-2}-\frac{4}{x+3}+\frac{4}{2-x}=0.$

25. $\frac{x-1}{x+1}+\frac{x+1}{x-1}=\frac{5x}{x^2-1}.$

26. $\frac{1}{x^2-3x}-\frac{1}{9-x^2}=\frac{13}{16x}.$

27. $\frac{1}{x^2-1}-\frac{1}{1-x}=\frac{7}{8}-\frac{1}{x+1}.$

28. $\frac{1}{x^2-3x}+\frac{1}{x^2+4x}=\frac{9}{2x^2}.$

29. $\frac{1}{3x-6}+\frac{7x}{72(x+2)}=\frac{5}{x^2-4}.$

30. $\frac{1}{x^2-4}-\frac{3}{2-x}=1+\frac{1}{3x+6}.$

31. $\frac{x}{x+1}+\frac{x+1}{x+2}=\frac{x-2}{x-1}+\frac{x-1}{x}.$

32. $\frac{1}{x}+\frac{1}{x+4}=\frac{1}{x+1}+\frac{1}{x+2}.$

33. $\frac{x}{x-3}-\frac{x-3}{x}+\frac{x}{x+3}-\frac{x+3}{x}=\frac{2}{3}.$

34. $\frac{x+3}{x+1}+\frac{x-6}{x-4}=\frac{x+4}{x+2}+\frac{x-5}{x-3}.$

35. $\frac{x-3}{x-4}-\frac{x-4}{x-5}=\frac{x-6}{x-7}-\frac{x-7}{x-8}.$

36. $\frac{x}{x-2}-\frac{x+1}{x-1}=\frac{x-8}{x-6}-\frac{x-9}{x-7}.$

37. $\frac{x+5}{x+4}-\frac{x-6}{x-7}=\frac{x-4}{x-5}-\frac{x-15}{x-16}.$

38. $\frac{x-7}{x-9}-\frac{x-9}{x-11}=\frac{x-13}{x-15}-\frac{x-15}{x-17}.$

39. $\frac{x+3}{x+6}-\frac{x+6}{x+9}=\frac{x+2}{x+5}-\frac{x+5}{x+8}.$

40. $\frac{x+2}{x}+\frac{x-7}{x-5}-\frac{x+3}{x+1}=\frac{x-6}{x-4}.$

41. $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}=\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}.$

42. $\frac{5x-8}{x-2}+\frac{6x-44}{x-7}-\frac{10x-8}{x-1}=\frac{x-8}{x-6}.$

43. $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1}.$

44. $\frac{25-\frac{1}{3}x}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=5+\frac{23}{x+1}$

45. $\frac{3}{4-2x}+\frac{30}{8(1-x)}=\frac{3}{2-x}+\frac{5}{2-2x}.$

46. $\frac{7}{x-4}-\frac{60}{5x-30}=\frac{10\frac{1}{2}}{3x-12}-\frac{8}{x-6}.$

47. $\frac{4}{x+3}-\frac{2}{x+1}=\frac{5}{2x+6}-\frac{2\frac{1}{2}}{2x+2}.$

48. $\frac{(2x-1)(3x+8)}{6x(x+4)}-1=0.$

49. $\frac{5x-64}{x-13}-\frac{2x-11}{x-6}=\frac{4x-55}{x-14}-\frac{x-6}{x-7}.$

50. $\frac{x-8}{x-10}+\frac{x-4}{x-6}=\frac{x-5}{x-7}+\frac{x-7}{x-9}.$

51. $x+\frac{1}{a}=a+\frac{1}{x}.$

52. $\frac{a}{a+x}+\frac{a}{a-x}=4.$

53. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{1}{a}+\frac{1}{b}.$

54. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{a}{b}+\frac{b}{a}.$

55. $\frac{x}{a}+\frac{a}{x}=\frac{b}{a}+\frac{a}{b}.$

56. $\frac{1}{x-a-b}=\frac{1}{x}-\frac{1}{a}-\frac{1}{b}.$

57. $\frac{a^2}{x-b}+\frac{b^2}{x-a}=a+b.$

58. $\frac{1}{x+a+b}=\frac{1}{x}+\frac{1}{a}+\frac{1}{b}.$

59. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{c+a}+\frac{1}{c+b}.$

60. $\frac{x}{x+a}+\frac{x}{x+b}=\frac{c}{c+a}+\frac{c}{c+b}.$

61. $\frac{x+a}{x-a}+\frac{x+b}{x-b}=\frac{x-a}{x+a}+\frac{x-b}{x+b}.$

62. $\frac{a}{x+a}+\frac{b}{x+b}=\frac{a-c}{x+a-c}+\frac{b+c}{x+b+c}.$

63. $\frac{1}{x+a+\frac{1}{x+b}}=\frac{1}{x-a+\frac{1}{x-b}}.$

64. $\frac{a+b}{x+b}+\frac{a+c}{x+c}=\frac{2(a+b+c)}{x+b+c}.$

65. $\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}.$

CHAPTER XVI.

PROBLEMS.

191. The *solving* of a problem by equations consists of three distinct parts:

(i.) The **Statement**; that is, the translation of the verbal statement of the problem into equations.

(ii.) The **Solving** of these equations.

(iii.) The **Discussion.** A problem may require for an answer a whole number, an arithmetical number, a real number, or numbers having some relation that is not expressed by the equations.

To state these and other such conditions of a problem, and to determine what solutions of the equations give answers to the problem, is called the *Discussion* of the problem.

The arithmetical numbers that will satisfy any equation are evidently the *arithmetical values* of its *positive* algebraic solutions.

EXAMPLE 1. Eleven times the number of persons in a room equals twice the square of that number increased by 12. How many are in the room?

Statement. Let $x =$ the *number* of persons;

then $$11x = 2x^2 + 12. \qquad (1)$$

Solving (1), we obtain $x = 4$ or $\frac{3}{2}$.

Discussion. The problem requires an arithmetical whole number as an answer; but the only such number that satisfies (1) is 4. Hence the one and only answer is 4 persons.

Example 2. A train travels 300 miles at a uniform rate; if the rate had been 5 miles an hour more, the journey would have taken 2 hours less; find the rate of the train.

Statement. Let $x =$ the *number* of miles travelled per hour;

then $300 \div x =$ the number of hours required for the journey,

and $300 \div (x + 5) =$ the number of hours the journey would have taken if the rate had been increased 5 miles an hour.

Hence, by the conditions of the problem, we have

$$\frac{300}{x} = \frac{300}{x + 5} + 2. \qquad (1)$$

Solving (1), we obtain $x = 25$ or -30.

Discussion. The number of miles per hour must be an arithmetical number that satisfies equation (1); hence the one and the only answer is 25 miles an hour.

Example 3. The square of the number of dollars a man is worth exceeds by 300 twenty times that number. How much is the man worth?

Statement. Let $x =$ the number of dollars the man is worth;

then $x^2 = 20\,x + 300.$ (1)

Solving (1), we obtain $x = 30$ or -10.

Discussion. If a debt be regarded as a negative possession, both of these solutions will give answers; that is, the man either has \$30 or owes \$10.

Example 4. The sum of the ages of a father and son is 100 years; and one-tenth of the product of their ages, in years, exceeds the father's age by 180. How old is each?

Statement. Let $x =$ the number of years in the father's age;

then $100 - x =$ the number of years in the son's age.

Hence $0.1\,x\,(100 - x) = x + 180.$ (1)

Solving (1), we obtain $x = 60$, and $100 - x = 40$,

or $x = 30$, and $100 - x = 70$.

Discussion. The father must be older than the son; hence the father must be 60, and the son 40, years old.

Both of the solutions of (1) would give answers if the problem read as follows: The sum of the ages of *two persons* is 100 years; and one-tenth of the product of their ages, in years, exceeds the age of one of them by 180. How old is each?

Example 5. Find a *real* number whose square increased by 13 is equal to 4 times the number.

Statement. Let $x =$ the number;

then $$x^2 + 13 = 4x. \qquad (1)$$

Solving (1), we obtain $$x = 2 \pm 3\sqrt{-1}. \qquad (2)$$

Discussion. Since there is no *real* number that will satisfy (1), the problem is impossible.

If the word "*real*" were omitted in the problem, both the values of x in (2) would be answers.

Example 6. A cistern can be filled by two pipes running together in $22\frac{1}{2}$ minutes; the larger pipe alone would fill the cistern in 24 minutes less than the smaller one. Find in what time each would fill it.

Statement. Suppose the larger pipe to fill the cistern in x minutes; then the smaller pipe will fill it in $x + 24$ minutes. Also, $1 \div x$ and $1 \div (x + 24)$ are the portions of the cistern which each pipe will fill in one minute, and $1 \div 22\frac{1}{2}$ is the portion that both together will fill in one minute.

Hence $$\frac{1}{x} + \frac{1}{x + 24} = \frac{1}{22\frac{1}{2}}. \qquad (1)$$

Solving (1), we obtain $x = 36$ or -15.

Discussion. The answer must be an arithmetical number, but 36 is the only such number that will satisfy (1). Hence the larger pipe would fill the cistern in 36 minutes, and the smaller one in $36 + 24$, or 60, minutes.

Exercise 64.

1. Find two numbers one of which is 4 times the other, and whose product is 196.

2. Find two numbers whose sum is 25, and whose product is 144.

3. Find two numbers whose sum is 15, and whose product is -250.

4. Divide 71 into two parts, the sum of the squares of which is 2561.

5. A rectangular court is 5 yards longer than it is broad; its area is 1886 square yards. Find its length and breadth.

6. The sum of the squares of two consecutive whole numbers is 1013. Find the numbers.

7. The sum of the reciprocals of two consecutive whole numbers is $\frac{15}{56}$. Find the numbers.

8. If a train travelled 5 miles an hour faster, it would take 1 hour less to travel 210 miles. Find the rate of the train.

9. The perimeter of a rectangular field is 500 yards, and its area is 14,400 square yards. Find the length of the sides.

10. The perimeter of one square exceeds that of another by 100 feet; and the area of the larger square exceeds 3 times the area of the smaller by 325 square feet. Find the length of their sides.

11. A lawn 50 feet long and 34 feet broad has a path of uniform width round it; the area of the path is 540 square feet. Find its width.

12. A man travels 108 miles, and finds that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled 2 miles an hour faster. At what rate did he travel?

13. The product of the sum and difference of a real number and its reciprocal is $3\frac{3}{4}$. Find the number.

14. A cistern can be filled by 2 pipes in $33\frac{1}{3}$ minutes. To fill the cistern, the larger pipe takes 15 minutes less than the smaller. Find in what time it will be filled by each pipe singly.

15. A hall can be paved with 200 square tiles of a certain size; if each tile were one inch longer each way, it would take 128 tiles. Find the size of the tile.

16. There are two square buildings paved with stones each a foot square. The side of one building exceeds that of the other by 12 feet, and the two pavements together contain 2120 stones. Find the sides of the buildings.

17. Find the number such that the product of the numbers obtained by adding to it 3 and 5 respectively is less by 1 than the square of its double.

18. The plate of a mirror is 18 inches by 12, and it is to be framed with a frame of uniform width, whose area is to be equal to that of the glass. Find the width of the frame.

19. A and B distribute $100 each in charity; A relieves 5 persons more than B, and B gives to each $1 more than A. How many did they each relieve?

20. The difference between the hypotenuse and two sides of a right-angle triangle is 3 and 6 respectively. Find the sides.

21. In the centre of a square garden is a square lawn; outside this is a gravel walk 4 feet wide, and then a flower border 6 feet wide. If the flower border and lawn together contain 721 square feet, find the area of the lawn.

22. What is the property of a person whose income is $ 2150, when he has $\frac{2}{3}$ of it invested at 4 per cent, $\frac{1}{4}$ at 3 per cent, and the remainder at 2 per cent?

23. A person bought a certain number of oxen for \$1200, and, after losing 3, sold the rest for \$20 a head more than they cost him, thus gaining \$35 by the bargain. How many oxen did he buy?

24. A can do a piece of work in 10 days; but after he has been upon it 4 days, B is sent to help him, and they finish it together in 2 days. In what time would B have done the whole work?

25. A and B can reap a field together in 7 days, which A alone could reap in 10 days. In what time could B alone reap it?

26. A can build a wall in 8 days, which A and B can do together in 5 days. How long would B take to do it alone?

27. A does $\frac{5}{9}$ of a piece of work in 10 days, when B comes to help him, and they take 3 days more to finish it. How long would B take to do it alone?

28. The tens' digit of a certain number exceeds the units' digit by 4, and when the number is divided by the sum of the digits, the quotient is 7. Find the number.

29. Find a number of three digits, each greater by 1 than that which follows it, so that its excess above $\frac{1}{4}$ of the number formed by inverting the digits shall be 36 times the sum of the digits.

30. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but on the enemy coming in sight, the front was increased by 845 men, and the whole was thus drawn up in 5 lines. Find the number of men.

31. The sum of two numbers is 14, and the quotient of the less divided by the greater is $\frac{9}{16}$ of the quotient of the greater divided by the less.

32. Find two fractions whose sum is $\frac{5}{6}$, and whose difference is equal to their product.

33. Two men start at the same time to meet each other from towns which are 25 miles apart. One takes 18 minutes longer than the other to walk a mile, and they meet in 5 hours. How fast does each walk?

Let x = the number of minutes it takes the first man to walk a mile.

34. A and B together can do a piece of work in a certain time. If they each did one-half of the work separately, A would have to work 1 day less, and B 2 days more than before. Find the time in which A and B together do the work.

35. A man bought a certain number of railway shares for $1875; he sold all but 15 of them for $1740, gaining $4 per share on their cost price. How many shares did he buy?

36. The denominator of a fraction exceeds the numerator by 4; and if 5 is taken from each, the sum of the reciprocal of the new fraction and 4 times the original fraction is 5. Find the original fraction.

37. A person swimming in a stream which runs $1\frac{1}{2}$ miles per hour, finds that it takes him 4 times as long to swim a mile up the stream as it does to swim the same distance down. At what rate does he swim?

38. What is the property of a person whose income is $1140, when one-twelfth of it is invested at 2 per cent, one-half at 3 per cent, one-third at $4\frac{1}{2}$ per cent, and the remainder pays him no dividend?

39. A person having 7 miles to walk increases his speed one mile an hour after the first mile, and is half an hour less on the road than he would have been had he not altered his rate. How long did it take to walk the 7 miles?

Let x miles an hour be his rate at first.

40. The diagonal and the longer side of a rectangle are together 5 times the shorter side, and the longer side exceeds the shorter by 35 yards. Find the area of the rectangle.

41. The price of photographs is raised 50 cents per dozen; and, in consequence, 4 less than before are sold for $ 5. Find the original price.

42. A boat's crew can row 8 miles an hour in still water. What is the speed of a river's current if it takes them 2 hours and 40 minutes to row 8 miles up and 8 miles down the river?

Let $x =$ the number of miles the current runs in an hour;

then $$\frac{8}{8+x}+\frac{8}{8-x}=\frac{8}{3}.$$

43. At a concert $ 300 was received for reserved seats, and the same amount for unreserved seats. A reserved seat cost 75 cents more than an unreserved seat, but 360 more tickets were sold for unreserved than for reserved seats. How many tickets were sold altogether?

44. Out of a cask containing 60 gallons of alcohol a certain quantity is drawn off and replaced by water. Of the mixture a second quantity, 14 gallons more than the first, is drawn off and replaced by water. The cask then contains as much water as alcohol. How much was drawn off the first time?

Let $x =$ the number of gallons drawn off the first time; then, in the first mixture,

$60 - x =$ the number of gallons of alcohol,

and $x =$ the number of gallons of water.

$$\therefore \frac{60-x}{60}(60-x-14)=\frac{x}{60}(60-x-14)+x+14.$$

45. A cyclist rode 180 miles at a uniform rate. If he had ridden 3 miles an hour slower than he did, it would have taken him 3 hours longer. How many miles an hour did he ride?

46. A man drives to a certain place at the rate of 8 miles an hour. Returning by a road 3 miles longer at the rate of 9 miles an hour, he takes $7\frac{1}{2}$ minutes longer than in going. How long is each road?

47. A father's age is equal to the united ages of his 5 children, and 5 years ago his age was double their united ages. How old is the father?

48. A and B are two stations 300 miles apart. Two trains start simultaneously from A and B, each to the opposite station. The train from A reaches B 9 hours, the train from B reaches A 4 hours, after they met. When did they meet, and what was the rate of each train?

49. If a carriage wheel $14\frac{2}{3}$ feet in circumference takes one second more to revolve, the rate of the carriage per hour will be $2\frac{2}{3}$ miles less. How fast is the carriage travelling?

Let x = number of miles travelled per hour; then

$$\frac{1}{x}+\frac{1}{10}=\frac{1}{x-2\frac{2}{3}}.$$

50. The number of square inches in the surface of a cubical block exceeds the number of inches in the sum of its edges by 288. Find its edge and volume.

51. A cistern can be filled by 2 pipes running together in 2 hours 55 minutes. The larger pipe by itself will fill it sooner than the smaller one by 2 hours. Find the time in which each pipe separately will fill it.

52. My gross income is \$3000. After paying the income tax, and then deducting from the remainder a percentage less by 1 than that of the income tax, the income is reduced to \$2736. Find the rate per cent of the income tax.

53. A set out from C towards D at the rate of 5 miles an hour. After he had gone 45 miles, B set out from D towards C, and went every hour $\frac{1}{28}$ of the entire distance. After travelling as many hours as he went miles in an hour, he met A. Find the distance from C to D.

192. Interest Formulas. In problems of interest, the quantities involved are the *principal*, *rate*, *time*, *interest*, and *amount*.

Let $\$p =$ the principal,
$\$r =$ the interest on \$1 for 1 year, or the given rate,
$t =$ the number of years in the time,
$\$i =$ the interest for the given time and rate,
$\$a =$ the amount; or sum of principal and interest.

Since the interest on \$1 for 1 year is $\$r$, the interest on $\$p$ for 1 year is $\$pr$, and therefore the interest on $\$p$ for t years is $\$prt$.

$$\therefore i = prt. \qquad (1)$$

Also $$a = p + prt; \qquad (2)$$

for $$\$a = \$p + \$i = \$p + \$prt.$$

If any three of the five numbers, a, i, p, r, t, be given, the other two may be found from formulas (1) and (2).

EXAMPLE. Find the principal that will amount to \$1584 in 5 years 4 months at 6 per cent.

Here $a = 1584$, $t = 5\frac{1}{3}$, $r = 0.06$. Substituting in (2), we have

$$1584 = p + p\,(0.06)(5\tfrac{1}{3}) = 1.32\,p.$$

$$\therefore p = 1584 \div 1.32 = 1200.$$

Hence the principal is \$1200.

Exercise 65.

1. Solve $i = prt$ for p, r, and t.

2. Solve $a = p + prt$ for p, r, and t.

3. Find the interest on $4760 for 4 years 6 months at 5½ per cent.

4. Find the amount of $3500 for 5 years 4 months at 6 per cent.

5. Find the interest on $7240 for 3 years 3 months at 8 per cent.

6. The interest on $1250 for 8 months is $50. Find the rate per cent.

7. The amount of $1050 for 2 years 6 months is $1260. Find the rate.

8. The interest on $3420 at 6 per cent is $649.80. Find the time.

9. A sum of money doubles in 12 years 6 months. Find the rate.

10. Find the principal that will yield $262.50 per month at 7 per cent.

11. Find the time in which $1350 will amount to $1809 at 6 per cent.

12. The interest in 4 years 3 months at 4 per cent is $2099.50. Find the principal.

13. Find the time in which a sum of money will double itself at 6 per cent.

14. The interest on $1270 for 8 months is $76.20. Find the rate.

15. At 4 per cent how much money is required to yield $2500 interest annually?

CHAPTER XVII.

SYSTEMS OF LINEAR EQUATIONS, AND PROBLEMS.

193. The **Degree** of an equation which contains two or more unknown letters is the degree of that term which is of the highest degree in these letters.

Thus $ax + by = 7$ is a linear equation; $ax^2 + by^2 = c$ and $2x + 3y = xy$ are quadratic equations; and $3x^3 + xy^2 = 4y$ and $axy^2 + 2bxy = c$ are cubic equations.

194. A single equation which contains two or more unknown letters is satisfied by an indefinite number of sets of values of these letters. Any such set of values of these letters is called a **Solution** of the equation.

Thus if in the equation

$$y = 3x + 2, \tag{1}$$

we put 1 for x, we obtain $y = 5$; hence one solution of this equation is $x = 1$, $y = 5$; another solution is $x = 2$, $y = 8$; another is $x = 3$, $y = 11$; and so on. In fact, we may give to x any value we please, and then find a corresponding value for y. Hence the equation has an infinite number of solutions.

Again, of the equation

$$y = 5x - 4, \tag{2}$$

one solution is $x = 1$, $y = 1$; another is $x = 2$, $y = 6$; another is $x = 3$, $y = 11$; and so on indefinitely.

Note that equations (1) and (2) have the solution $x = 3$, $y = 11$ in common.

195. Equivalent Equations. (For definition see § 98.) The principles concerning the equivalency of equations with one unknown letter hold true for equations with two or more unknown letters. For convenience of reference we here restate these principles.

I. *If for any expression we substitute an identical expression, the resulting equation will be equivalent to the first.*

II. *If the same expression be added to, or subtracted from, both members of an equation, the resulting equation will be equivalent to the first.*

III. *If both members of an equation be multiplied by the same known expression, the resulting equation will be equivalent to the first.*

IV. *If both members of an integral equation be multiplied by the same unknown integral expression, new solutions will be introduced.*

Thus multiplying both members of any integral equation by $x - y + 2$ will introduce the solutions of the equation $x - y + 2 = 0$.

V. *If both members of a fractional equation be multiplied by the denominator of any fractional term, in general no new solution will be introduced.*

VI. *If the square root of both members of an equation be extracted, and the double sign ± be written in one member, the two resulting equations will be equivalent to the given equation.*

VII. *If both members of an equation be raised to the same power, in general, new solutions will be introduced.*

Thus by squaring both members of the equation

$$x - 4 = 2y, \tag{1}$$

we obtain

$$(x - 4)^2 = 4y^2. \tag{2}$$

Now by VI., (2) is equivalent to the two equations

$$x - 4 = 2y \text{ and } x - 4 = -2y.$$

Hence the solutions of $x - 4 = -2y$ were introduced by squaring both members of (1).

196. Independent Equations. Any two equations that are not equivalent are said to be *Independent.* Independent equations express different relations between their unknown letters, while equivalent equations express the same relation. In general, two or more independent equations involving the same unknown letters have one or more solutions in common.

Thus the two equations

$$y = 3x + 2 \tag{1}$$

and

$$y = 5x - 4 \tag{2}$$

are *independent;* for one cannot be obtained from the other by any of the principles of § 195. As shown in § 194, these equations have the solution $x = 3$, $y = 11$ in common.

197. A **System** of equations is a group of two or more independent equations which are to be satisfied by the same set, or sets, of values of their unknown letters. Any such set of values of these letters is called a **Solution** of the system. The equations of a system are said to be **Simultaneous.**

Thus equations (1) and (2) in § 196 form a system of equations of which a solution is $x = 3$ and $y = 11$.

198. Elimination is the process of combining two or more equations of a system so as to obtain another equation with one less unknown letter. The letter that is caused to disappear is said to be *eliminated.*

There are in general use two methods of elimination:

(i.) By *Addition* or *Subtraction.*

(ii.) By *Substitution* or *Comparison.*

199. Elimination by Addition or Subtraction.

Example 1. Solve the system

$$\left.\begin{array}{lr} 3x+7y=27, & (1) \\ 5x+2y=16. & (2) \end{array}\right\} (a)$$

Multiply (1) by 5, $15x+35y=135.$ (3)

Multiply (2) by 3, $15x+6y=48.$ (4)

Subtract (4) from (3), $29y=87.$

$\therefore\ y=3.$ (5)

Multiply (5) by 7, $7y=21.$ (6)

Subtract (6) from (1), $3x=6.$

$\therefore\ x=2.$ (7)

By multiplying (1) by 5 and (2) by 3, we obtained two equations (3) and (4) with the terms in x equal; hence by subtracting the one from the other, we obtained an equation without x; that is, we *eliminated* x.

If system (a) has any solution, it is $x=2$, $y=3$.

Proof. To verify that $x=2$, $y=3$ is a solution of system (a), we put 2 for x and 3 for y in equations (1) and (2); the results obtained are the identities,

$$3\times 2+7\times 3=27,$$

and

$$5\times 2+2\times 3=16;$$

hence $x=2$, $y=3$ is a solution of system (a).

Example 2. Solve the system

$$\left.\begin{array}{lr} 7x+2y=47, & (1) \\ 5x-4y=1. & (2) \end{array}\right\} (a)$$

Multiply (1) by 2, $14x+4y=94.$ (3)

Add (2) and (3), $19x=95.$

$\therefore\ x=5.$ (4)

Multiply (4) by 7, $7x=35.$ (5)

Subtract (5) from (1), $2y=12$, or $y=6.$ (6)

Hence if system (a) has any solution, it is $x=5$, $y=6$.

Proof. Substituting 5 for x and 6 for y in equations (1) and (2), we obtain the identities,

$$7\times 5+2\times 6=47,$$

and

$$5\times 5-4\times 6=1;$$

hence $x=5$, $y=6$ is a solution of system (a).

Hence, to eliminate by addition or subtraction, we have the following rule:

Multiply each equation by such a number as will make the coefficients of one unknown letter numerically equal in the resulting equations.

Add these resulting equations, or subtract one from the other, according as the numerically equal coefficients are unlike or like in quality.

EXAMPLE 3. Solve

$$\left.\begin{aligned}171x-213y&=642, &(1)\\ 114x-326y&=244. &(2)\end{aligned}\right\}(a)$$

Multiply (1) by 2, $342x-426y=1284.$ (3)

Multiply (2) by 3, $342x-978y=732.$ (4)

Subtract (4) from (3), $552y=552.$

$\therefore\ y=1.$ (5)

Multiply (5) by 326, $326y=326.$ (6)

Add (6) and (2), $114x=570.$

$\therefore\ x=5.$ (7)

The student should verify that $x=5$, $y=1$ is a solution of (a).

NOTE. It is generally best to eliminate that unknown letter first which has the smaller coefficients in the two equations.

Exercise 66.

Solve the system

1. $\left.\begin{aligned}3x+4y&=10\\ 4x+y&=9\end{aligned}\right\}$

2. $\left.\begin{aligned}x+2y&=13\\ 3x+y&=14\end{aligned}\right\}$

3. $\left.\begin{aligned}2x-y&=9\\ 3x-7y&=19\end{aligned}\right\}$

4. $\left.\begin{aligned}4x+7y&=29\\ x+3y&=11\end{aligned}\right\}$

5. $\left.\begin{aligned}2x+y&=10\\ 7x+8y&=53\end{aligned}\right\}$

6. $\left.\begin{aligned}5x+6y&=17\\ 6x+5y&=16\end{aligned}\right\}$

7. $\left.\begin{aligned}8x-y&=34\\ x+8y&=53\end{aligned}\right\}$

8. $\left.\begin{aligned}15x+7y&=29\\ 9x+15y&=39\end{aligned}\right\}$

9. $$\left.\begin{aligned} 14x - 3y &= 39 \\ 6x + 17y &= 35 \end{aligned}\right\}$$

10. $$\left.\begin{aligned} 28x - 23y &= 33 \\ 63x - 25y &= 101 \end{aligned}\right\}$$

11. $$\left.\begin{aligned} 35x + 17y &= 86 \\ 56x - 13y &= 17 \end{aligned}\right\}$$

12. $$\left.\begin{aligned} 5x - 7y &= 0 \\ 7x + 5y &= 74 \end{aligned}\right\}$$

13. $$\left.\begin{aligned} 15x + 77y &= 92 \\ 55x - 33y &= 22 \end{aligned}\right\}$$

14. $$\left.\begin{aligned} 5x &= 7y - 21 \\ 21x - 9y &= 75 \end{aligned}\right\}$$

15. $$\left.\begin{aligned} 6y - 5x &= 18 \\ 12x - 9y &= 0 \end{aligned}\right\}$$

16. $$\left.\begin{aligned} 21x - 50y &= 60 \\ 28x - 27y &= 199 \end{aligned}\right\}$$

200. Equivalent Systems. Two systems of equations are said to be *equivalent* when they have the same solutions; that is, when every solution of either system is a solution of the other.

Thus the systems

$$\left.\begin{aligned} 3x + 7y &= 27, \ (1) \\ 5x + 2y &= 16, \ (2) \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} 8x + 9y &= 43, \ (3) \\ 5x + 2y &= 16, \ (4) \end{aligned}\right\}$$

are equivalent; for each system has the solution $x = 2$, $y = 3$, and neither has any other solution.

Note that equation (3) can be obtained by adding (1) and (2).

201. *If for any equation of a system an equivalent equation be substituted, the resulting system will be equivalent to the given system.*

Thus the systems

$$\left.\begin{aligned} 2x + y &= 5, \ (1) \\ x - 3y &= 2, \ (2) \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} 6x + 3y &= 15, \ (3) \\ 2x - 6y &= 4, \ (4) \end{aligned}\right\}$$

are equivalent; for, (3) being equivalent to (1), and (4) to (2), the solutions common to (1) and (2) will be common also to (3) and (4).

202. *If the equation obtained by adding or subtracting any two equations, or their equivalent equations, be substituted for either one of them, the resulting system will be equivalent to the given system. That is, if* $A = 0$ *and*

B = 0 *be any two equations in* x *and* y, *then the systems* (*a*) *and* (*b*) *will be equivalent.*

$$\left.\begin{aligned} A &= 0, \ (1) \\ B &= 0, \ (2) \end{aligned}\right\} (a) \qquad \left.\begin{aligned} A &= 0, \ (3) \\ mA + nB &= 0, \ (4) \end{aligned}\right\} (b)$$

For any solution of system (*a*) reduces A and B each to 0, and therefore satisfies both (3) and (4); hence any solution of system (*a*) is a solution of (*b*).

Again, any solution of (*b*) reduces A to 0, and therefore reduces equation (4) to $nB = 0$, or $B = 0$; hence any solution of (*b*) is a solution of (*a*).

Hence (*a*) and (*b*) are equivalent systems.

The same reasoning would apply if (1) and (2) contained any number of unknown letters, or if the sign + in (4) were changed to −.

The letters m and n denote any *known* expressions by which (1) and (2) are multiplied before they are added to obtain (4).

From this principle it follows that any set of values of x and y obtained by elimination by addition or subtraction will be a solution of the given system.

203. *Any system of two linear equations in* x *and* y *has, in general, one and only one solution.*

By the principles of equivalent equations, any system of two linear equations can be reduced to an equivalent system of the form

$$\left.\begin{aligned} ax + by &= c, \ (1) \\ a'x + b'y &= c'. \ (2) \end{aligned}\right\} (a)$$

Multiply (1) by b', $\qquad ab'x + bb'y = b'c. \qquad (3)$

Multiply (2) by b, $\qquad a'bx + bb'y = bc'. \qquad (4)$

Subtract (4) from (3), $\quad (ab' - a'b)x = b'c - bc'. \qquad (5)$

$$\therefore x = \frac{b'c - bc'}{ab' - a'b}. \qquad (6)$$

Multiply (1) by a', $$aa'x + a'by = a'c. \tag{7}$$

Multiply (2) by a, $$aa'x + ab'y = ac'. \tag{8}$$

Subtract (8) from (7), $$(a'b - ab')y = a'c - ac'. \tag{9}$$

$$\therefore\ y = \frac{a'c - ac'}{a'b - ab'}. \tag{10}$$

By §§ 202 and 195, (6) and (1) form a system equivalent to (a); hence the value of x in system (a) is given in (6). For like reason the value of y in system (a) is given in (10). Hence the set of values of x and y in (6) and (10) is the one, and only, solution of system (a).

Example 1. Solve
$$\left.\begin{aligned} 51x - 14y &= 287, \quad (1)\\ 14x - 51y &= -157. \quad (2)\end{aligned}\right\}(a)$$

Since the coefficients of x and y are simply interchanged in the two equations, by addition we obtain an equation in which the coefficients of x and y are equal. Hence we proceed as follows:

Add (1) and (2), $$65x - 65y = 130.$$

Divide by 65, $$x - y = 2. \tag{3}$$

Multiply (3) by 14, $$14x - 14y = 28. \tag{4}$$

Subtract (4) from (1), $$37x = 259.$$

$$\left.\begin{aligned} \therefore\ x &= 7. \quad (5)\\ \text{Subtract (3) from (5),}\qquad y &= 5. \quad (6)\end{aligned}\right\}(b)$$

By § 202, system (a) is equivalent to system (1) and (3); system (1) and (3) is equivalent to system (3) and (5); and system (3) and (5) is equivalent to system (b); hence the solution of (a) is given in (b).

Example 2. Solve
$$\left.\begin{aligned} (x-1)(y-2)-(x-2)(y-1) &= -2, \quad (1)\\ (x+2)(y+2)-(x-2)(y-2) &= 32. \quad (2)\end{aligned}\right\}(a)$$

$$\left.\begin{aligned} \text{From (1),}\qquad -x + y &= -2. \quad (3)\\ \text{From (2),}\qquad 4x + 4y &= 32. \quad (4)\end{aligned}\right\}(b)$$

Multiply (3) by 4, $$-4x + 4y = -8. \tag{5}$$

$$\left.\begin{array}{lll}\text{Add (4) and (5),} & 8y = 24, \text{ or } y = 3. & (6)\\ \text{Subtract (6) from (3),} & -x = -5, \text{ or } x = 5. & (7)\end{array}\right\}(c)$$

By § 201 system (*b*) is equivalent to system (*a*), and by § 202 (6) and (3) form a system equivalent to either (*b*) or (*c*); hence the solution of (*a*) is given in (*c*).

EXAMPLE 3. Solve

$$\left.\begin{array}{ll} 3x - \dfrac{y-5}{7} = \dfrac{4x-3}{2}, & (1)\\[2ex] \dfrac{3y+4}{5} - \dfrac{2x-5}{3} = y. & (2)\end{array}\right\}(a)$$

Clearing (1) and (2) of fractions, and transposing, we obtain

$$\left.\begin{array}{lll}\text{from (1),} & 14x - 2y = -31, & (3)\\ \text{from (2),} & 10x + 6y = 37. & (4)\end{array}\right\}(b)$$

$$\begin{array}{lll}
\text{Multiply (3) by 3,} & 42x - 6y = -93. & (5)\\
\text{Add (4) and (5),} & 52x = -56, \text{ or } x = -\tfrac{14}{13}. & (6)\\
\text{Multiply (3) by 5,} & 70x - 10y = -155. & (7)\\
\text{Multiply (4) by 7,} & 70x + 42y = 259. & (8)\\
\text{Subtract (7) from (8),} & 52y = 414, \text{ or } y = \tfrac{207}{26}. & (9)
\end{array}$$

The solution of system (*a*) is given in (6) and (9).

Exercise 67.

Solve the system

1. $\left.\begin{aligned}57x + 25y &= 3772\\ 25x + 57y &= 1148\end{aligned}\right\}.$

2. $\left.\begin{aligned}93x + 15y &= 123\\ 15x + 93y &= 201\end{aligned}\right\}$

3. $\left.\begin{aligned}15x + 19y &= 18\\ 19x + 15y &= 50\end{aligned}\right\}$

4. $\left.\begin{aligned}\frac{x}{2} + \frac{y}{3} &= 1\\ \frac{x}{4} - \frac{2y}{3} &= 3\end{aligned}\right\}$

5. $\left.\begin{aligned}\frac{x}{5} + 5y &= -4\\ \frac{y}{5} + 5x &= 4\end{aligned}\right\}$

6. $\left.\begin{aligned}\frac{x+2}{3} + 4y &= 2\\ \frac{y+11}{11} - \frac{x+1}{2} &= 1\end{aligned}\right\}$

7. $\left.\begin{aligned}\frac{2x+3y}{5} + \frac{y+6}{7} &= 2\\ \frac{2x-5y}{3} + \frac{x+7}{4} &= 1\end{aligned}\right\}$

8. $$\left.\begin{array}{l}\dfrac{x-2}{3}-\dfrac{y+2}{4}=0\\[2ex] \dfrac{2x-5}{5}-\dfrac{11-2y}{7}=0\end{array}\right\}$$

9. $$\left.\begin{array}{l}\dfrac{x-2}{3}-\dfrac{y+5}{2}=0\\[2ex] \dfrac{2x-7}{3}-\dfrac{13-y}{16}=0\end{array}\right\}$$

10. $$\left.\begin{array}{l}\dfrac{x}{2}-\dfrac{y}{5}\;=4\\[2ex] \dfrac{x}{7}+\dfrac{y}{15}=3\end{array}\right\}$$

11. $$\left.\begin{array}{l}2x+y\;\;=0\\ \frac{1}{2}y-3x=8\end{array}\right\}$$

12. $$\left.\begin{array}{r}\frac{1}{7}x+\frac{1}{5}y=1\frac{3}{7}\\ x+\frac{1}{3}y=4\frac{2}{3}\end{array}\right\}$$

13. $$\left.\begin{array}{l}3x-7y=0\\ \frac{2}{7}x+\frac{5}{3}y=7\end{array}\right\}$$

14. $$\left.\begin{array}{l}\frac{1}{5}x-\frac{1}{4}y=\;\;0\\ 3x+\frac{1}{2}y=17\end{array}\right\}$$

15. $$\left.\begin{array}{l}ax+by=(a+b)^2\\ ax-by=\;a^2-b^2\end{array}\right\}$$

16. $$\left.\begin{array}{l}ax+by=a^2+b^2\\ bx+ay=2ab\end{array}\right\}$$

17. $$\left.\begin{array}{l}ax+by=a^2-b^2\\ bx+ay=a^2-b^2\end{array}\right\}$$

18. $$\left.\begin{array}{l}x+\;\;y=\;a+b\\ ax-by=b^2-a^2\end{array}\right\}$$

19. $$\left.\begin{array}{l}b^2x-a^2y=0\\ bx+ay\;=a+b\end{array}\right\}$$

20. $$\left.\begin{array}{l}x-\;\;y=a-b\\ ax-by=2a^2-2b^2\end{array}\right\}$$

21. $$\left.\begin{array}{l}ax-by=a^2+b^2\\ x+\;\;y=2a\end{array}\right\}$$

22. $$\left.\begin{array}{l}bx-ay=b^3\\ ax-by=a^3\end{array}\right\}$$

23. $$\left.\begin{array}{l}ax+by=1\\ bx+ay=1\end{array}\right\}$$

24. $$\left.\begin{array}{l}(a+b)x-(a-b)y=3ab\\ (a+b)y-(a-b)x=ab\end{array}\right\}$$

25. $$\left.\begin{aligned} a^2x + b^2y &= c^2 \\ a^3x + b^3y &= c^3 \end{aligned}\right\}$$

26. $$\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= \frac{1}{ab} \\ \frac{x}{a'} - \frac{y}{b'} &= \frac{1}{a'b'} \end{aligned}\right\}$$

27. $$\left.\begin{aligned} \frac{3x}{a} + \frac{2y}{b} &= 3 \\ \frac{9x}{a} - \frac{6y}{b} &= 3 \end{aligned}\right\}$$

28. $$\left.\begin{aligned} qx - rb &= p(a - y) \\ \frac{qx}{a} + r &= p\left(1 + \frac{y}{b}\right) \end{aligned}\right\}$$

29. $$\left.\begin{aligned} \frac{x}{m} + \frac{y}{m'} &= 1 \\ \frac{x}{m'} - \frac{y}{m} &= 1 \end{aligned}\right\}$$

30. $$\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 \\ \frac{x}{3a} + \frac{y}{6b} &= \frac{2}{3} \end{aligned}\right\}$$

31. $$\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 2 \\ \frac{x}{a'} &= \frac{y}{b'} \end{aligned}\right\}$$

32. $$\left.\begin{aligned} \frac{x}{a} - \frac{y}{b} &= 1 \\ \frac{x}{b} + \frac{y}{a} &= \frac{a}{b} \end{aligned}\right\}$$

204. Elimination by Substitution.

Example. Solve
$$\left.\begin{array}{lr} 3x + 5y = 19, & (1) \\ 5x - 4y = 7. & (2) \end{array}\right\} (a)$$

Solve (1) for x, $$x = \tfrac{1}{3}(19 - 5y). \qquad (3)$$

Substitute this value of x in (2), and we have
$$\tfrac{5}{3}(19 - 5y) - 4y = 7.$$

Multiply by 3, $$95 - 25y - 12y = 21.$$

$$\left.\begin{array}{lr} \therefore\ y = 2. & (4) \\ x = 3. & (5) \end{array}\right\} (b)$$

Put 2 for y in (3), $x = 3$.

Putting 2 for y and 3 for x in (1) and (2), we obtain identities; hence the solution of system (a) is given in (b).

Hence, to eliminate by substitution, we have the rule:

From one of the given equations find the value of one of the unknown letters in terms of the other, and substitute this value for that letter in the other equation.

Exercise 68.

Eliminating by substitution, solve the system

1. $\left.\begin{array}{l}7x+4y=1\\9x+4y=3\end{array}\right\}$

2. $\left.\begin{array}{l}x-11y=1\\111y-9x=99\end{array}\right\}$

3. $\left.\begin{array}{l}3x+5y=19\\5x-4y=7\end{array}\right\}$

4. $\left.\begin{array}{l}8x-21y=5\\6x+14y=-26\end{array}\right\}$

5. $\left.\begin{array}{r}3x-11y=0\\19x-19y=8\end{array}\right\}$

6. $\left.\begin{array}{r}8x-11y=0\\25x-17y=139\end{array}\right\}$

7. $\left.\begin{array}{l}\dfrac{x}{3}-\dfrac{y}{6}=\dfrac{1}{2}\\[2ex]\dfrac{x}{5}-\dfrac{3y}{10}=\dfrac{1}{2}\end{array}\right\}$

8. $\left.\begin{array}{l}\frac{1}{3}x+3y+14=0\\\frac{1}{5}x+5y+4=0\end{array}\right\}$

9. $\left.\begin{array}{l}\frac{1}{3}(7+x)=\frac{1}{5}(9+y)\\\frac{1}{7}(11+x+y)=\frac{1}{5}(9+y)\end{array}\right\}$

10. $\left.\begin{array}{l}\frac{1}{2}(x+1)=\frac{1}{3}(y+2)\\\frac{1}{4}(x+y)=\frac{1}{3}(y+2)\end{array}\right\}$

11. $\left.\begin{array}{l}(x+1)(y+5)=(x+5)(y+1)\\xy+x+y=(x+2)(y+2)\end{array}\right\}$

12. $\left.\begin{array}{l}xy-(x-1)(y-1)=6(y-1)\\x-y=1\end{array}\right\}$

205. *If* (*a*) *be any system of equations in which* A *does not contain* x, *and* (*b*) *a system obtained from* (*a*) *by substituting* A *for* x *in equation* (2); *then the systems* (*a*) *and* (*b*) *will be equivalent.*

$$\left.\begin{array}{ll}x=A, & (1)\\B=C. & (2)\end{array}\right\}(a)\qquad\qquad\left.\begin{array}{ll}x=A, & (1)\\B'=C'. & (2')\end{array}\right\}(b)$$

For any solution of system (a) will evidently satisfy (2) after A has been substituted for its equal x; hence any solution of system (a) will be a solution of (b).

Again, any solution of system (b) will evidently satisfy (2′) after x has been substituted for its equal A; hence any solution of system (b) will be a solution of system (a).

Therefore systems (a) and (b) are equivalent.

The same reasoning would apply if systems (a) and (b) each contained three or more equations.

From this principle it follows that any set of values of x and y obtained by eliminating by substitution will be a solution of the given system.

Thus in the example of § 204, system (a) is equivalent to the system (2) and (3); and, by § 205, this system is equivalent to the system (3) and (4), which, by § 205, is equivalent to (b).

Example. Solve

$$\left.\begin{aligned} 2x - 3y &= 1, \qquad (1)\\ 5x + 2y &= 126. \qquad (2)\end{aligned}\right\} (a)$$

Solve (1) for x, and solve (2) for x,

$$\left.\begin{aligned} x &= \frac{3y+1}{2}. \qquad (3)\\ x &= \frac{126-2y}{5}. \qquad (4)\end{aligned}\right\} (b)$$

Substitute in (4) the value of x given in (3), and we have

$$\frac{3y+1}{2} = \frac{126-2y}{5}. \qquad (5)$$

Put 13 for y in (3),

$$\left.\begin{aligned} \therefore\ y &= 13. \qquad (6)\\ x &= 20. \qquad (7)\end{aligned}\right\} (c)$$

By § 201 system (a) is equivalent to (b); and by § 205 equations (3) and (6) form a system equivalent to either system (b) or (c). Hence (a) and (c) are equivalent systems.

Note. Since in (5) we compare the two values of x given in (3) and (4), this special form of the method of elimination by substitution is sometimes called *Elimination by Comparison.*

Exercise 69.

Eliminating by substitution, or comparison, solve

1. $$\left.\begin{aligned}\frac{2x}{3}+y&=16\\ x+\frac{y}{4}&=14\end{aligned}\right\}$$

2. $$\left.\begin{aligned}x-y&=5\\ \frac{x}{4}-\frac{y}{5}&=2\end{aligned}\right\}$$

3. $$\left.\begin{aligned}\frac{x}{9}+\frac{y}{7}&=10\\ \tfrac{1}{8}x+y&=50\end{aligned}\right\}$$

4. $$\left.\begin{aligned}x&=3y\\ \tfrac{1}{8}x+y&=34\end{aligned}\right\}$$

5. $$\left.\begin{aligned}\frac{5x}{6}-y&=3\\ x-\frac{5y}{6}&=8\end{aligned}\right\}$$

6. $$\left.\begin{aligned}\frac{x}{5}+\frac{y}{2}&=5\\ x-y&=4\end{aligned}\right\}$$

7. $$\left.\begin{aligned}\tfrac{2}{5}x-\tfrac{1}{12}y&=3\\ 4x-y&=20\end{aligned}\right\}$$

8. $$\left.\begin{aligned}\frac{x}{3}+\frac{y}{4}&=0\\ 3x-7y&=37\end{aligned}\right\}$$

9. $$\left.\begin{aligned}\frac{x+1}{10}&=\frac{3y-5}{2}\\ \frac{x+1}{10}&=\frac{x-y}{8}\end{aligned}\right\}$$

10. $$\left.\begin{aligned}\frac{x+3}{5}&=\frac{8-y}{4}\\ \frac{3(x+y)}{8}&=\frac{x+3}{5}\end{aligned}\right\}$$

206. Systems of Fractional Equations. In solving a system of fractional equations that involve the reciprocals of x and y, it is better to regard these reciprocals as the unknown numbers. Clearing such equations of fractions would introduce new solutions.

EXAMPLE 1. Solve

$$\left.\begin{aligned}\frac{a}{x}+\frac{c}{y}&=m, &(1)\\ \frac{b}{x}+\frac{d}{y}&=n. &(2)\end{aligned}\right\}(a)$$

Multiply (1) by b, $$\frac{ab}{x}+\frac{bc}{y}=bm. \qquad (3)$$

Multiply (2) by a, $$\frac{ab}{x}+\frac{ad}{y}=an. \qquad (4)$$

Subtract (4) from (3), $$\frac{bc-ad}{y}=bm-an.$$

$$\therefore\ y=\frac{bc-ad}{bm-an}. \qquad (5)$$

Multiply (1) by d, $$\frac{ad}{x}+\frac{cd}{y}=dm. \qquad (6)$$

Multiply (2) by c, $$\frac{bc}{x}+\frac{cd}{y}=cn. \qquad (7)$$

Subtract (7) from (6), $$\frac{ad-bc}{x}=dm-cn.$$

$$\therefore\ x=\frac{ad-bc}{dm-cn}. \qquad (8)$$

Hence (8) and (5) give the only solution of system (a).

Clearing (1) and (2) of fractions, we obtain

$$\left.\begin{aligned} ay+cx&=mxy, && (9)\\ \text{and}\qquad by+dx&=nxy. && (10)\end{aligned}\right\}(b)$$

By inspection we see that one solution of system (b) is $x=0$, $y=0$; hence this solution was introduced by clearing (1) and (2) of fractions.

Example 2. Solve

$$\left.\begin{aligned} \frac{1}{4x}+\frac{1}{3y}&=2, && (1)\\ \frac{1}{y}-\frac{1}{2x}&=1. && (2)\end{aligned}\right\}(a)$$

$$\left.\begin{aligned} &\text{Multiply (1) by 12,} & \frac{3}{x}+\frac{4}{y}&=24. && (3)\\ &\text{Multiply (2) by 2,} & \frac{2}{y}-\frac{1}{x}&=2. && (4)\end{aligned}\right\}(b)$$

Multiply (4) by 3, $$\frac{6}{y}-\frac{3}{x}=6. \qquad (5)$$

$$\left.\begin{aligned} &\text{Add (3) and (5),} & \frac{10}{y}&=30. \quad \therefore\ y=\tfrac{1}{3}.\\ &\text{Put } \tfrac{1}{3} \text{ for } y \text{ in (4),} & 6-\frac{1}{x}&=2. \quad \therefore\ x=\tfrac{1}{4}.\end{aligned}\right\}(c)$$

To put system (a) in the form of (b), we multiply each equation by the L. C. M. of the coefficients of x and y in that equation.

Exercise 70.

Solve the system

1. $$\left.\begin{aligned}\frac{8}{x}-\frac{9}{y}&=1\\ \frac{10}{x}+\frac{6}{y}&=7\end{aligned}\right\}$$

2. $$\left.\begin{aligned}\frac{9}{x}-\frac{4}{y}&=2\\ \frac{18}{x}+\frac{18}{y}&=10\end{aligned}\right\}$$

3. $$\left.\begin{aligned}\frac{6}{x}-\frac{5}{y}&=9\\ \frac{7}{x}-\frac{2}{y}&=5\end{aligned}\right\}$$

4. $$\left.\begin{aligned}\frac{3}{x}+\frac{4}{y}&=3\\ \frac{6}{x}-\frac{2}{y}&=1\end{aligned}\right\}$$

5. $$\left.\begin{aligned}\frac{5}{x}+\frac{16}{y}&=79\\ \frac{16}{x}-\frac{1}{y}&=44\end{aligned}\right\}$$

6. $$\left.\begin{aligned}\frac{6}{x}-\frac{7}{y}&=2\\ \frac{2}{x}+\frac{14}{y}&=3\end{aligned}\right\}$$

7. $$\left.\begin{aligned}\frac{5}{3x}+\frac{2}{5y}&=7\\ \frac{7}{6x}-\frac{1}{10y}&=3\end{aligned}\right\}$$

8. $$\left.\begin{aligned}\frac{1}{2x}+\frac{2}{3y}&=3\\ \frac{3}{4x}+\frac{4}{5y}&=3.9\end{aligned}\right\}$$

9. $$\left.\begin{aligned}\frac{m}{x}+\frac{n}{y}&=a\\ \frac{b}{x}+\frac{c}{y}&=d\end{aligned}\right\}$$

10. $$\left.\begin{aligned}\frac{m}{x}-\frac{n}{y}&=\frac{mr}{n}\\ \frac{n}{x}-\frac{m}{y}&=\frac{nr}{m}\end{aligned}\right\}$$

11. $$\left.\begin{aligned}\frac{2}{mx}-\frac{1}{ny}&=3\\ \frac{3}{mx}+\frac{4}{ny}&=7\end{aligned}\right\}$$

12. $$\left.\begin{aligned}\frac{7}{ax}+\frac{3}{by}&=2\\ \frac{5}{ax}-\frac{2}{by}&=7\end{aligned}\right\}$$

13. $$\left.\begin{aligned}\frac{3}{ax}-\frac{2}{by}&=4\\ \frac{b}{x}+\frac{a}{y}&=7\end{aligned}\right\}$$

14. $$\left.\begin{aligned}\frac{m}{nx}+\frac{n}{my}&=m+n\\ \frac{n}{x}+\frac{m}{y}&=m^2+n^2\end{aligned}\right\}$$

207. Systems of Three or More Equations. To solve a system of three equations in x, y, and z, we may eliminate one unknown letter, as z, between the first and second equations, then eliminate z between the first and third, or the second and third, equations. We shall thus obtain a system of two equations in x and y which can be solved by either of the previous methods.

In like manner from a system of four equations we can obtain a system of three equations; and so on.

EXAMPLE 1. Solve

$$\left.\begin{aligned}6x+2y-5z&=13, &&(1)\\ 3x+3y-2z&=13, &&(2)\\ 7x+5y-3z&=26, &&(3)\end{aligned}\right\}(a)$$

First, eliminate y between (1) and (2), and then between (1) and (3).

$$\left.\begin{aligned}&\text{Multiply (1) by 3,} && 18x+6y-15z=39.\\ &\text{Multiply (2) by 2,} && 6x+6y-4z=26.\\ &\text{Subtract,} && 12x-11z=13. && (4)\\ &\text{Multiply (1) by 5,} && 30x+10y-25z=65.\\ &\text{Multiply (3) by 2,} && 14x+10y-6z=52.\\ &\text{Subtract,} && 16x-19z=13. && (5)\end{aligned}\right\}(b)$$

Solving system (b), we obtain

$$\left.\begin{aligned}z&=1, && (6)\\ x&=2. && (7)\end{aligned}\right\}(c)$$

From (6), (7), and (1), $y=3$. (8)

The systems (b) and (c) are equivalent.

But (b) with (1) forms a system equivalent to (a); hence (c) with (1), or (c) with (8), forms a system equivalent to (a). Hence the solution of system (a) is $x=2$, $y=3$, $z=1$.

Example 2. Solve

$$\left.\begin{array}{rr} 3x + 2y + 4z = 19, & (1) \\ 2x + 5y + 3z = 21, & (2) \\ 3x - y + z = 4. & (3) \end{array}\right\} (a)$$

From (3), $$y = 3x + z - 4, \qquad (4)$$

Substituting in (1) and (2) the value of y in (4), we obtain

$$3x + 2(3x + z - 4) + 4z = 19,$$

and $$2x + 5(3x + z - 4) + 3z = 21;$$

or

$$\left.\begin{array}{r} 9x + 6z = 27, \\ 17x + 8z = 41. \end{array}\right\} (b)$$

and

Solving system (b), we obtain

$$\left.\begin{array}{rr} z = 3, & (5) \\ x = 1. & (6) \end{array}\right\} (c)$$

From (4), (5), (6), $$y = 2. \qquad (7)$$

By § 205 system (b) with (4) forms a system equivalent to (a); hence (c) with (4), or (c) with (7), forms a system equivalent to (a). Hence the solution of (a) is $x = 1$, $y = 2$, $z = 3$.

Example 3. Solve

$$\left.\begin{array}{rr} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 36, & (1) \\ \frac{1}{x} + \frac{3}{y} - \frac{1}{z} = 28, & (2) \\ \frac{1}{x} + \frac{1}{3y} + \frac{1}{2z} = 20. & (3) \end{array}\right\} (a)$$

Multiply (3) by 6, $$\frac{6}{x} + \frac{2}{y} + \frac{3}{z} = 120.$$

Multiply (2) by 6, $$\frac{6}{x} + \frac{18}{y} - \frac{6}{z} = 168.$$

Subtract, $$\frac{16}{y} - \frac{9}{z} = 48. \qquad (4)$$

Subtract (1) from (2), $$\frac{2}{y} - \frac{2}{z} = -8. \qquad (5)$$

(4) and (5) form system (b).

Solving system (b), we obtain

$$\left.\begin{array}{rr} z = \frac{1}{16}, & (6) \\ y = \frac{1}{12}. & (7) \end{array}\right\} (c)$$

From (1), (6), and (7), $$x = \tfrac{1}{8}. \qquad (8)$$

Exercise 71.

Solve

1. $\left.\begin{aligned} x+3y+4z&=14\\ x+2y+z&=7\\ 2x+y+2z&=2\end{aligned}\right\}$

2. $\left.\begin{aligned} x+2y+2z&=11\\ 2x+y+z&=7\\ 3x+4y+z&=14\end{aligned}\right\}$

3. $\left.\begin{aligned} 3x-2y+z&=2\\ 2x+3y-z&=5\\ x+y+z&=6\end{aligned}\right\}$

4. $\left.\begin{aligned} x+y+z&=1\\ 2x+3y+z&=4\\ 4x+9y+z&=16\end{aligned}\right\}$

5. $\left.\begin{aligned} 5x+3y+7z&=2\\ 2x-4y+9z&=7\\ 3x+2y+6z&=3\end{aligned}\right\}$

6. $\left.\begin{aligned} x+2y-3z&=6\\ 2x+4y-7z&=9\\ 3x-y-5z&=8\end{aligned}\right\}$

7. $\left.\begin{aligned} x-2y+3z&=2\\ 2x-3y+z&=1\\ 3x-y+2z&=9\end{aligned}\right\}$

8. $\left.\begin{aligned} 3x+2y-z&=20\\ 2x+3y+6z&=70\\ x-y+6z&=41\end{aligned}\right\}$

9. $\left.\begin{aligned} 2x+3y+4z&=20\\ 3x+4y+5z&=26\\ 3x+5y+6z&=31\end{aligned}\right\}$

10. $\left.\begin{aligned} 3x-4y&=6z-16\\ 4x-y&=z+5\\ x&=3y+2(z-1)\end{aligned}\right\}$

11. $\left.\begin{aligned} ax+by&=1\\ by+cz&=1\\ cz+ax&=1\end{aligned}\right\}$

12. $\left.\begin{aligned} cy+bz&=bc\\ az+cx&=ca\\ bx+ay&=ab\end{aligned}\right\}$

13. $\left.\begin{aligned} x-\frac{y}{5}&=6\\ y-\frac{z}{7}&=8\\ z-\frac{x}{2}&=10\end{aligned}\right\}$

14. $\left.\begin{aligned} \tfrac{1}{2}(x+z-5)&=y-z\\ \tfrac{1}{2}(x+z-5)&=2x-11\\ 2x-11&=9-(x+2z)\end{aligned}\right\}$

15. $\left.\begin{aligned} x+20&=\tfrac{3}{2}y+10\\ x+20&=2z+5\\ 2z+5&=110-(y+z)\end{aligned}\right\}$

16. $$\left.\begin{aligned}\frac{1}{x}-\frac{2}{y}+4&=0\\ \frac{1}{y}-\frac{1}{z}+1&=0\\ \frac{2}{z}+\frac{3}{x}&=14\end{aligned}\right\}$$

17. $$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=36\\ \frac{1}{x}+\frac{3}{y}-\frac{1}{z}&=28\\ \frac{1}{x}+\frac{1}{3y}+\frac{1}{2z}&=20\end{aligned}\right\}$$

18. $$\left.\begin{aligned}\frac{1}{2x}+\frac{1}{4y}-\frac{1}{3z}&=\frac{1}{4}\\ \frac{1}{x}&=\frac{1}{3y}\\ \frac{1}{x}-\frac{1}{5y}+\frac{4}{z}&=2\tfrac{2}{15}\end{aligned}\right\}$$

PROBLEMS LEADING TO SYSTEMS OF EQUATIONS.

208. If a problem contains just as many independent conditions as there are unknown numbers, the statement of each condition will be an independent equation; hence the statement of the whole problem will be a system of equations.

Example 1. Find two numbers such that twice the greater exceeds three times the less by 6, and that twice the less exceeds the greater by 2.

Let $x =$ the greater number, and $y =$ the less.

Then, by the conditions of the problem, we have

$$2x-3y=6, \qquad (1)$$

and $$2y-x=2. \qquad (2)$$

Multiply (2) by 2, $4y-2x=4.$ (3)

Add (1) and (3), $y=10.$

Put 10 for y in (1), $2x-30=6$, or $x=18$.

Hence the two numbers are 18 and 10.

Example 2. A number of two digits is equal to six times the sum of its digits, and the digit in the tens' place is greater by one than the digit in the units' place. Find the number.

Let $x =$ the digit in tens' place,

and $y =$ the digit in units' place.

Then $x + y =$ the sum of the digits,

and $10x + y =$ the number.

Hence, by the conditions of the problem, we have

$$\left.\begin{aligned} 10x + y &= 6(x + y), &(1)\\ \text{and}\quad x - y &= 1. &(2)\end{aligned}\right\}$$

From (1), $4x - 5y = 0.$

Multiply (2) by 4, $4x - 4y = 4.$

$$\left.\begin{aligned}\text{Subtract,}\quad y &= 4.\\ \text{Add (2) and (3),}\quad x &= 5.\end{aligned}\right\}$$

Hence the number is 54.

Example 3. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to $\frac{5}{8}$; and if the numerator and denominator are each diminished by 1, it becomes equal to $\frac{1}{2}$. Find the fraction.

Let $x =$ the numerator, and $y =$ the denominator; then

$$\left.\begin{aligned}\frac{x+2}{y+1} &= \frac{5}{8}, &(1)\\ \text{and}\quad \frac{x-1}{y-1} &= \frac{1}{2}. &(2)\end{aligned}\right\}(a)$$

The solution of system (a) is $x = 8$, $y = 15$; hence the fraction is $\frac{8}{15}$.

Example 4. A man and a boy can do in 15 days a piece of work which would be done in 2 days by 7 men and 9 boys. How long would it take one boy to do it?

Let $x =$ the number of days it would take one man to do the whole work; and $y =$ the number of days it would take one boy to do it.

Then in a day a man would do $1 \div x$ of the work, and a boy $1 \div y$ of it.

$$\left.\begin{aligned}&\text{Hence} && \frac{1}{x}+\frac{1}{y}=\frac{1}{15}, && (1)\\ &\text{and} && \frac{7}{x}+\frac{9}{y}=\frac{1}{2} && (2)\end{aligned}\right\}(a)$$

Equation (1) states the condition that a man and a boy together would do $\frac{1}{15}$ of the whole work in a day.

Equation (2) states the condition that 7 men and 9 boys together would do one-half the whole work in a day.

The solution of system (a) is $x = 20$, $y = 60$.

Hence one man would do the work in 20 days, and one boy in 60 days.

Exercise 72.

1. Six horses and 7 cows can be bought for $1250, and 13 cows and 11 horses can be bought for $2305. Find the value of each animal.

2. Four times B's age exceeds A's age by 20 years, and $\frac{1}{3}$ of A's age is less than B's age by 2 years. Find their ages.

3. Find a fraction such that if 1 be added to its denominator it reduces to $\frac{1}{2}$, and if 2 be added to its numerator it reduces to $\frac{4}{5}$.

4. A man being asked his age, replied: "If you take 2 years from my present age the result will be double my wife's age, and 3 years ago her age was $\frac{1}{3}$ of what mine will be in 12 years." Find their ages.

5. One-eleventh of A's age is greater by 2 years than $\frac{1}{7}$ of B's, and twice B's age is equal to what A's age was 13 years ago. Find their ages.

6. In 8 hours A walks 12 miles more than B does in 7 hours; and in 13 hours B walks 7 miles more than A does in 9 hours. How many miles does each walk per hour?

7. At an election the majority was 162, which was $\frac{3}{11}$ of the whole number of voters. What was the number of the votes on each side?

8. A and B have $250 between them; but if A were to lose half his money, and B $\frac{3}{8}$ of his, they would then have only $100. How much has each?

9. A man bought 8 cows and 50 sheep for $1125. He sold the cows at a profit of 20%, and the sheep at a profit of 10%, and received in all $1287.50. What was the cost of each cow and of each sheep?

10. Twenty-eight tons of goods are to be carried in carts and wagons, and it is found that this will require 15 carts and 12 wagons, or else 24 carts and 8 wagons. How much can each cart and each wagon carry?

11. A and B can perform a certain task in 30 days, working together. After 12 days, however, B was called off, and A finished it by himself 24 days after. How long would each take to do the work alone?

12. Find the fraction such that if you quadruple the numerator and add 3 to the denominator the fraction will be doubled, but if you add 2 to the numerator and quadruple the denominator, the fraction will be halved.

13. The first edition of a book had 600 pages, and was divided into two parts. In the second edition $\frac{1}{4}$ of the second part was omitted and 30 pages were added to the first part. The change made the two parts of the same length. How many pages were in each part in the first edition?

14. A marketman bought eggs, some at 3 for 5 cents, and some at 4 for 5 cents, and paid for all $5.60; he afterwards sold them at 24 cents a dozen, clearing $1.80. How many eggs did he buy at each price?

15. In a bag containing black and white balls, half the number of white is equal to a third of the number of black; and twice the whole number of balls exceeds 3 times the number of black balls by 4. How many balls does the bag contain?

16. A crew that can row 10 miles an hour down a river, finds that it takes twice as long to row up the river as to row down. Find the rate of the current.

17. A certain number between 10 and 100 is 8 times the sum of its digits, and if 45 be subtracted from it the digits will be reversed. Find the number.

18. If A were to receive $50 from B, he would then have twice as much as B would have left; but if B were to receive $50 from A, B would have 3 times as much as A would have left. How much has each?

19. A farmer sold 30 bushels of wheat and 50 bushels of barley for $93.75. He also sold at the same prices 50 bushels of wheat and 30 bushels of barley for $96.25. What was the price of the wheat per bushel?

20. One rectangle is of the same area as another which is 6 yards longer and 4 yards narrower; it is also of the same area as a third, which is 8 yards longer and 5 yards narrower. What is the area of each?

21. A boy rows 8 miles with the current in 1 hour 4 minutes, and returns against the current in $2\frac{2}{7}$ hours. At what rate would he row in still water, and what is the rate of the current?

22. A, B, C, D, have $1450 between them; A has twice as much as C, and B has 3 times as much as D; also C and D together have $250 less than A. Find how much each has.

23. A, B, C, D have $1350 between them; A has 3 times as much as C, and B 5 times as much as D; also A and B together have $ 250 less than 8 times what C has. Find how much each has.

24. A number consists of 2 digits followed by zero. If the digits be interchanged, the number will be diminished by 180; if the left-hand digit be halved, and the other digit be interchanged with zero, the number will be diminished by 454. Find the number.

25. A train travelled a certain distance at a uniform rate; had the speed been 6 miles an hour more, the journey would have occupied 4 hours less; and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more. Find the distance.

Let $x =$ the number of miles the train runs per hour,

and $y =$ the number of hours the journey takes.

Then
$$\left.\begin{aligned} xy &= (x+6)(y-4), \\ \text{and}\quad xy &= (x-6)(y+6). \end{aligned}\right\}$$

26. A traveller walks a certain distance; had he gone $\frac{1}{2}$ mile an hour faster, he would have walked it in $\frac{4}{5}$ of the time; had he gone $\frac{1}{2}$ mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance.

27. A man walks 35 miles, partly at the rate of 4 miles an hour, and partly at 5; if he had walked at 5 miles an hour when he walked at 4, and *vice versa*, he would have covered 2 miles more in the same time. Find the time he was walking.

28. A fishing-rod consists of two parts; the length of the upper part is $\frac{5}{7}$ that of the lower part; and 9 times the upper part together with 13 times the lower part exceeds 11 times the whole rod by 36 inches. Find the lengths of the two parts.

29. A man put $12,000 at interest in three sums, the first at 5 per cent, the second at 4 per cent, and the third at 3 per cent, receiving for the whole $490 a year. The sum at 5 per cent is half as much as the other two sums. Find each of the three sums.

30. A, B, and C can together do a piece of work in 30 days; A and B can together do it in 32 days; B and C can together do it in 120 days. Find the time in which each alone could do the work.

31. A certain company in a tavern found, when they came to pay their bills, that if there had been 3 more persons to pay the same bill, they would have paid $1 each less than they did; and if there had been 2 fewer persons, they would have paid $1 each more than they did. Find the number of persons, and the number of dollars each paid.

32. A railway train, after travelling 1 hour, is detained 30 minutes, after which it proceeds at $\frac{5}{4}$ of its former rate, and arrives 20 minutes late. If the detention had occurred 10 miles further on, the train would have arrived 5 minutes later than it did. Find the first rate of the train, and the distance travelled.

Let $x =$ the number of miles the train at first ran per hour;

and $y =$ the number of miles in the whole distance travelled.

Then $y - x =$ the number of miles to be travelled after the detention,

$\frac{y - x}{x} =$ the number of hours required to travel $y - x$ miles at the rate before the detention,

and $\frac{4(y - x)}{5x} =$ the number of hours required to travel $y - x$ miles at the rate after the detention.

$$\left.\begin{aligned} &\text{Hence} && \frac{y-x}{x}-\frac{4(y-x)}{5x}=\frac{10}{60}. && (1)\\ &\text{Similarly,} && \frac{y-x-10}{x}-\frac{4(y-x-10)}{5x}=\frac{5}{60}. && (2)\end{aligned}\right\}$$

Subtract (2) from (1),
$$\frac{10}{x}-\frac{40}{5x}=\frac{5}{60}.$$
$$\therefore x=24,\ y=44.$$

Hence the first rate was 24 miles an hour, and the distance travelled was 44 miles.

33. A railway train, after travelling 1 hour, meets with an accident which delays it 1 hour, after which it proceeds at $\frac{3}{5}$ of its former rate, and arrives at the terminus 3 hours behind time; had the accident occurred 50 miles further on, the train would have arrived 1 hour 20 minutes sooner. Find the length of the line, and the original rate of the train. *Ans.* 100 miles, 25 miles per hour.

34. A jockey has 2 horses and 2 saddles. The saddles are worth \$15 and \$10 respectively. The value of the better horse and better saddle is $\frac{4}{3}$ that of the other horse and saddle; and the value of the better saddle and poorer horse is $\frac{13}{15}$ that of the other horse and saddle. Find the worth of each horse.

35. Five thousand dollars is divided between A, B, C, and D. B gets half as much as A; the excess of C's share over D's share is equal to $\frac{1}{3}$ of A's share, and if B's share were increased by \$500 he would have as much as C and D have between them. Find how much each gets.

36. A party was composed of a certain number of men and women, and, when 4 of the women were gone, it was observed that there were left just half as many men again as women; they came back, however, with their husbands, and now there were only a third as many men again as women. What were the original numbers of each?

37. Two vessels contain mixtures of wine and water; in one there is 3 times as much wine as water, in the other 5 times as much water as wine. Find how much must be drawn off from each to fill a third vessel which holds 7 gallons, in order that its contents may be half wine and half water.

38. There is a number of 3 digits, the last of which is double the first; when the number is divided by the sum of the digits, the quotient is 22; and when by the product of the last two, 11. Find the number.

39. Some smugglers found a cave which would exactly hold the cargo of their boat; viz. 13 bales of silk and 33 casks of rum. While unloading, a revenue cutter came in sight, and they were obliged to sail away, having landed only 9 casks and 5 bales, and filled $\frac{1}{3}$ of the cave. How many bales separately, or how many casks, would it contain?

40. There are 2 alloys of silver and copper, of which one contains twice as much copper as silver, and the other 3 times as much silver as copper. How much must be taken from each to weigh a kilogram, of which the silver and the copper shall be equal in weight?

41. A person rows a distance of 20 miles, and back again, in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the time of his going and returning.

42. A and B can do a piece of work in m days, A and C can do the same piece in n days, and B and C can do it in p days. Find in how many days each can do the work.

43. For $26.25 we can buy either 32 pounds of tea and 15 pounds of coffee, or 36 pounds of tea and 9 pounds of coffee. Find the price of a pound of each.

44. A pound of tea and 3 pounds of sugar cost $1.50; but if sugar were to rise 50 per cent, and tea 10 per cent, they would cost $1.75. Find the price of tea and sugar.

45. A person possesses a certain capital which is invested at a certain rate per cent. A second person has $5000 more capital than the first person, and invests it at 1 per cent more; thus his income exceeds that of the first person by $400. A third person has $7500 more capital than the first, and invests it at 2 per cent more; thus his income exceeds that of the first person by $750. Find the capital of each person and the rate at which it is invested.

46. Two plugs are opened in the bottom of a cistern containing 192 gallons of water; after 3 hours one of the plugs becomes stopped, and the cistern is emptied by the other in 11 more hours; had 6 hours occurred before the stoppage, it would have required only 6 hours more to empty the cistern. How many gallons will each plug-hole discharge in an hour, supposing the discharge uniform?

47. A certain number of persons were divided into 3 classes, such that the majority of the first and second classes together over the third was 10 less than 4 times the majority of the second and third together over the first; but if the first class had 30 more, and the second and third together 29 less, the first would have outnumbered the last 2 classes by 1. Find the number in each class when the whole number was 34 more than 8 times the majority of the third class over the second.

48. Two persons, A and B, could finish a work in m days; they worked together n days, when A was called off, and B finished it in p days. In what time could each do it?

49. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; if the circumference of the fore-wheel be increased by $\frac{1}{4}$ of its present size, and the circumference of the hind-wheel by $\frac{1}{5}$ of its present size, the 6 will be changed to 4. Required the circumference of each wheel.

50. A hare is pursued by a greyhound, and is 60 of her own leaps before him. The hare takes 3 leaps in the time that the greyhound takes 2; but the greyhound goes as far in 3 leaps as the hare does in 7. In how many leaps will the greyhound catch the hare?

CHAPTER XVIII.

RATIO AND PROPORTION.

209. The **Ratio** of one number to another is the quotient of the first divided by the second.

The ratio of a to b is written $\frac{a}{b}$, $a \div b$, or $a : b$, either of which forms may be read 'a *is to* b' or 'a *by* b.' The ratio of 8 to 2 is $\frac{8}{2}$ or 4; the ratio of 7 to 5 is $\frac{7}{5}$.

The dividend is often called the **Antecedent,** and the divisor the **Consequent** of the ratio.

210. Since a ratio is a fraction, all the principles of fractions apply to ratios in whatever form they may be written.

Thus $$a : b = am : bm, \text{ or } \frac{a}{b} = \frac{am}{bm}; \qquad \S\ 83$$

$$a : b = (a \div m) : (b \div m), \text{ or } \frac{a}{b} = \frac{a \div m}{b \div m}; \qquad \S\ 83$$

$$(a : b)(c : d) = ac : bd, \text{ or } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}. \qquad \S\ 75$$

Two ratios may be compared by reducing them as fractions to equivalent fractions having a common denominator.

Example. Which is the greater ratio, 3 : 11 or 5 : 19?

Reducing the fractions $\frac{3}{11}$ and $\frac{5}{19}$ to a common denominator, we obtain

$$\frac{3}{11} = \frac{57}{209} \text{ and } \frac{5}{19} = \frac{55}{209};$$

hence the ratio $3 : 11 >$ the ratio $5 : 9$.

211. Ratios are *compounded* by taking their product.

Thus the ratio compounded of $3:4$ and $a:b$ is $3a:4b$.

The ratio $a^2:b^2$ is called the **Duplicate ratio** of $a:b$.
The ratio $a^3:b^3$ is called the **Triplicate ratio** of $a:b$.
The ratio $\sqrt{a}:\sqrt{b}$ is called the **Sub-duplicate ratio** of $a:b$.

212. The **Inverse** of a ratio is its reciprocal.

Thus the *inverse* of the ratio $a:b$ is the ratio $b:a$ (§ 84).

213. Incommensurable Numbers. Numbers whose ratio *can be exactly* expressed by two whole numbers are said to be **Commensurable.** Numbers whose ratio can *not* be *exactly* expressed by two whole numbers are said to be **Incommensurable.** The ratio of two incommensurable numbers is called an **Incommensurable Ratio.**

Any two fractions are commensurable, if their numerators and denominators are commensurable.

Thus the ratio of $\frac{3}{4}$ to $\frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{21}{20}$, or $21:20$;

and in general the ratio of $\frac{a}{b}$ to $\frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$, or $ad:bc$.

The surds $\sqrt{2}$ and $\sqrt{3}$ are incommensurable numbers, so also are $\sqrt{2}$ and 1; for we cannot find any fraction which is exactly equal to $\sqrt{2}$ or to $\sqrt{3}$. We can, however, find a fraction that will approximate the value of $\sqrt{2}$ or $\sqrt{3}$ as nearly as we please. Hence we can always find a commensurable ratio that will approximate as nearly as we please to any incommensurable ratio.

214. Proportion. Four numbers, a, b, c, d, are said to be in proportion, when the ratio of a to b is equal to the ratio of c to d.

The proportion is written in the form

$$\frac{a}{b} = \frac{c}{d},\ a:b = c:d, \text{ or } a:b::c:d,$$

and is read 'a *is to* b *as* c *is to* d,' or 'a *by* b *equals* c *by* d.'

When four numbers are proportional, the first and the fourth are called the **Extremes**, and the second and the third the **Means.**

Thus a and d are the extremes, and b and c are the means, of the proportion $a:b=c:d$.

215. *If four numbers are proportional, the product of the extremes is equal to the product of the means.*

For suppose $a:b=c:d$, or $\frac{a}{b}=\frac{c}{d}$.

Clear of fractions, $ad=bc$.

Hence, when three terms of a proportion are given, the fourth may be found. Thus if the first three terms of a proportion are x^2, $2a$, and $5b$, the fourth $=2a\times 5b\div x^2$.

216. Conversely, *if the product of two numbers is equal to the product of two other numbers, two of them may be made the extremes and the other two the means of a proportion.*

For, let $ad=bc$.

Divide by dc, $a:c=b:d$, or $b:d=a:c$.

Or divide by ab, $d:b=c:a$, or $c:a=d:b$.

217. If the product of the extremes of a proportion is equal to the product of its means, by § 216 we know that the proportion is true.

This principle furnishes a very simple and useful test of the truth of a proportion.

Thus $a:b=c:d$, $a:c=b:d$,

$b:a=d:c$, $b:d=a:c$,

are all true proportions, if only $ad=bc$.

218. *If* $$a : b = c : d, \qquad (1)$$

then a, b, c, d *are in proportion by*

(i.) *Inversion; that is,* $b : a = d : c.$ (2)

(ii.) *Alternation; that is,* $a : c = b : d.$ (3)

(iii.) *Addition; that is,* $a + b : b = c + d : d.$ (4)

(iv.) *Subtraction; that is,* $a - b : b = c - d : d.$ (5)

(v.) *Addition and Subtraction; that is,*

$$a + b : a - b = c + d : c - d. \qquad (6)$$

For, from (1), we have, by § 215,

$$ad = bc. \qquad (7)$$

Hence (2) and (3) are true proportions; for the product of the extremes in each equals the product of the means.

Adding $\pm bd$ to both members of (7), we have

$$ad \pm bd = bc \pm bd.$$

Hence (4) and (5) are true proportions; for the product of the extremes in each equals the product of the means.

Dividing the ratios in (4) by the corresponding ratios in (5), we obtain (6).

219. Continued Proportion. Numbers are said to be in *continued proportion* when the first is to the second as the second is to the third, and so on. Thus a, b, c, d are in continued proportion when $a : b = b : c = c : d$.

If $a : b = b : c$, then b is called a **Mean Proportional** *between* a and c, and c is called a **Third Proportional** *to* a and b.

If $a:b=b:c=c:d$, then b and c are called the **Two Mean Proportionals** between a and d.

220. *The mean proportional between two numbers equals the square root of their product.*

For if $a:b=b:c$, then $b^2=ac$, or $b=\sqrt{ac}$.

221. *The products of the corresponding terms of two or more proportions are proportional.*

That is, if $a:b=c:d, \quad (1)$

$e:f=g:h, \quad (2)$

and $m:n=r:s; \quad (3)$

then $aem:bfn=cgr:dhs. \quad (4)$

For multiplying together the corresponding members of equations (1), (2), and (3), by Axiom 3 we obtain (4).

222. *Like powers or like roots of proportionals are proportional.*

For if $a:b=c:d, \quad (1)$

then $(a:b)^n=(c:d)^n.$ § 121

$\therefore a^n:b^n=c^n:d^n.$ § 109

Also $\sqrt[n]{a:b}=\sqrt[n]{c:d}.$ § 121

$\therefore \sqrt[n]{a}:\sqrt[n]{b}=\sqrt[n]{c}:\sqrt[n]{d}.$ § 125

223. If $a:b=c:d$, then

(i.) $ma:mb=nc:nd;$

(ii.) $ma:nb=mc:nd.$

The proof is left as an exercise for the student.

224. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any one antecedent is to its consequent.*

For assume $a : b = c : d = e : f = \cdots,$

and let $a : b = r;$ then $c : d = r,\ e : f = r,\ \cdots.$

Hence $a = br,\ c = dr,\ e = fr,\ \cdots.$

Adding these equalities, we obtain

$$a + c + e + \cdots = (b + d + f + \cdots)r.$$

$$\therefore\ \frac{a + c + e + \cdots}{b + d + f + \cdots} = r = \frac{a}{b} = \frac{c}{d} = \cdots.$$

225. It is often very convenient to represent the value of a ratio by a single letter as in the last article. The following are additional examples.

Example 1. Given $a : b = c : d$, prove that

$$a^2 + ab : c^2 + cd = b^2 - 2\,ab : d^2 - 2\,cd. \qquad (1)$$

Let $a \div b = r$; then $c \div d = r.$

Hence $a = br$, and $c = dr.$

Substituting these values of a and c in each ratio, we have

$$\frac{a^2 + ab}{c^2 + cd} = \frac{b^2r^2 + b^2r}{d^2r^2 + d^2r} = \frac{b^2\,(r^2 + r)}{d^2\,(r^2 + r)} = \frac{b^2}{d^2},$$

and $$\frac{b^2 - 2\,ab}{d^2 - 2\,cd} = \frac{b^2 - 2\,b^2r}{d^2 - 2\,d^2r} = \frac{b^2\,(1 - 2\,r)}{d^2\,(1 - 2\,r)} = \frac{b^2}{d^2}.$$

Hence (1) is a true proportion, for its ratios are equal.

Example 2. Given $a : b = c : d = e : f$, prove that

$$a^3 + c^3 + e^3 : b^3 + d^3 + f^3 = ace : bdf. \qquad (1)$$

Let $a \div b = r$; then $c \div d = r$, and $e \div f = r.$

Hence $a = br,\ c = dr$, and $e = fr.$

Substituting the values of a, c, and e first in the product of the extremes, and then in the product of the means, we obtain

$$(a^3 + c^3 + e^3)bdf = (b^3 + d^3 + f^3)r^3bdf,$$

and

$$(b^3 + d^3 + f^3)ace = (b^3 + d^3 + f^3)r^3bdf.$$

Hence (1) is a true proportion; for the product of its extremes is equal to the product of its means.

Exercise 73.

1. Write the duplicate ratio of $2x : 3y$.

Find the ratio compounded of

2. The duplicate ratio of $4 : 3$, and the ratio $25 : 8$.

3. The ratio $32 : 27$, and the triplicate ratio of $3 : 2$.

4. The sub-duplicate ratio of $25 : 36$, and the ratio $6 : 7$.

5. Arrange the ratios $5 : 6$, $7 : 8$, $41 : 48$, and $31 : 36$ in descending order of magnitude.

6. For what value of x will the ratio $15 + x : 17 + x$ equal $\frac{1}{2}$?

7. What number must be added to each of the terms of the ratio $3 : 4$ to make it equal to the ratio $25 : 32$?

Let $x =$ the number to be added; then

$$\frac{3 + x}{4 + x} = \frac{25}{32}.$$

8. Find two numbers in the ratio of 5 to 6, and whose sum is 121.

9. Two numbers are in the ratio of 3 to 8, and the sum of their squares is 3577; find them.

Let $3x =$ the first number;
then $8x =$ the second number; for $3x : 8x = 3 : 8$.

10. The ages of two persons are as $3 : 4$, and 30 years ago they were as $1 : 3$; find their present ages.

Find the fourth proportional to

11. a, ab, c. **12.** $a^2, 2ab, 3b^2$. **13.** $x^3, xy, 5x^2y$.

Find the third proportional to

14. a^2b, ab. **15.** $x^3, 2x^2$. **16.** $3x, 6xy$. **17.** $1, x$.

Find a mean proportional between

18. a^2, b^2. **20.** $12ax^2, 3a^3$.

19. $2x^3, 8x$. **21.** $27a^2b^3, 3b$.

If $a : b = c : d$, show that

22. $ac : bd = c^2 : d^2$.

23. $a^2 : c^2 = a^2 - b^2 : c^2 - d^2$.

24. $2a + 3c : 3a + 2c = 2b + 3d : 3b + 2d$.

25. $la + mb : pa + gb = lc + md : pc + gd$.

26. $a : a + c = a + b : a + b + c + d$.

27. $a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$.

28. $a + b : c + d = \sqrt{a^2 + b^2} : \sqrt{c^2 + d^2}$.

29. $\sqrt{a^2 + b^2} : \sqrt{c^2 + d^2} = \sqrt[3]{a^3 + b^3} : \sqrt[3]{c^3 + d^3}$.

30. $a^2c + ac^2 : b^2d + bd^2 = (a + c)^3 : (b + d)^3$.

31. $\sqrt[n]{a^n + b^n} : \sqrt[n]{c^n + d^n} = \sqrt[r]{a^r - b^r} : \sqrt[r]{c^r - d^r}$.

32. If $a : b = b : c$, prove that $a : c = a^2 : b^2$.

33. If $a : b = b : c = c : d$, prove that $a : d = a^3 : b^3$.

Let $r = a \div b$; then $a = br$, $b = cr$, $c = dr$.

$\therefore abc = bcdr^3$. $\therefore a \div d = r^3 = a^3 \div b^3$.

34. If a, b, c, d be any four numbers, find what number must be added to each to make the results proportional.

35. The sides of a triangle are as $3:4:5$, and the perimeter is 480 yards; find the sides.

36. Divide the number 14 into two such parts that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 16 to 9.

37. Show that the ratio of any two fractions, not involving surds, can be expressed by the ratio of two whole numbers.

38. Express the ratio of $5\frac{1}{2}$ to $7\frac{4}{18}$ by the ratio of two whole numbers.

39. Express the ratio of $17\frac{1}{4}$ to $14\frac{3}{8}$ by the ratio of two whole numbers.

40. The sum of two numbers is 8, and their product is to the sum of their squares as 3 to 10. What are the numbers?

41. The sum of two numbers is 10, and the sum of their squares is to the square of their sum as 13 to 25. What are the numbers?

CHAPTER XIX.

THEORY OF EXPONENTS.

226. A base with any exponent is called an **Exponential Expression.**

Thus a^2, x^a, and b^{a+x} are exponential expressions.

227. *The* s*th root of the* r*th power of a number is equal to the* r*th power of its* s*th root.*

For let $$\sqrt[s]{a} = c; \qquad (1)$$

then $$a = c^s.$$

$$\therefore a^r = (c^s)^r = c^{rs}.$$

$$\therefore \sqrt[s]{a^r} = c^{rs \div s} = c^r. \qquad (2)$$

From (1), $$(\sqrt[s]{a})^r = c^r. \qquad (3)$$

From (2) and (3), $$\sqrt[s]{a^r} = (\sqrt[s]{a})^r. \qquad (4)$$

NOTE. Equality (4) is not strictly true except with the limitation in § 120. For example, $\sqrt[2]{a^4} = \pm a^2$, and $(\sqrt[2]{a})^4 = a^2$; hence $\sqrt[2]{a^4} = (\sqrt[2]{a})^4$ only when $\sqrt[2]{a^4}$ is restricted to its positive value.

228. Laws of Exponents. If a, b, m, and n denote any numbers, the five laws of exponents may be expressed in symbols as follows:

$$a^m \times a^n = a^{m+n}. \qquad (I.)$$

$$a^m \div a^n = a^{m-n}. \qquad (II.)$$

$$(a^m)^n = a^{mn}. \qquad (III.)$$

$$(ab)^m = a^m b^m. \qquad \text{(IV.)}$$

$$(a \div b)^m = a^m \div b^m. \qquad \text{(V.)}$$

229. The laws of exponents have all been proved for positive integral exponents. It now remains to show that these laws hold true for fractional and negative exponents; that is, are universally true. We *assume* that the fundamental law of exponents, (I.) of § 228, holds true for fractional and negative exponents, and assign to them the meaning which this law requires.

Thus to find the meaning of $a^{\frac{1}{3}}$, we have by law (I.)

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^{\frac{3}{3}} = a^1 = a;$$

that is, $a^{\frac{1}{3}}$ = one of the three equal factors of $a = \sqrt[3]{a}$.

Again, $a^{\frac{3}{2}} \cdot a^{\frac{3}{2}} = a^{\frac{3}{2}+\frac{3}{2}} = a^{\frac{3}{2}\times 2} = a^3;$

that is, $a^{\frac{3}{2}}$ = one of the two equal factors of $a^3 = \sqrt{a^3}$.

Thus $8^{\frac{1}{3}} = \sqrt[3]{8} = 2;\ 4^{\frac{3}{2}} = \sqrt[2]{4^3} = \sqrt[2]{64} = 8.$

230. *To find the Meaning of any Positive Fractional Exponent.*

Let r and s denote any positive integers; then assuming law (I.), we have

$$a^{\frac{r}{s}} \cdot a^{\frac{r}{s}} \cdots \text{to } s \text{ factors} = a^{\frac{r}{s}+\frac{r}{s}+\cdots \text{ to } s \text{ terms}}$$

$$= a^{\frac{r}{s}\cdot s} = a^r;$$

that is, $a^{\frac{r}{s}} = \sqrt[s]{a^r}$ or its equal $(\sqrt[s]{a})^r$. § 227

Hence a **Positive Fractional Exponent** denotes a root of a power, or a power of a root, of its base. The denominator indicates the root, and the numerator the power.

Thus $a^{\frac{3}{4}} = \sqrt[4]{a^3}$ or $(\sqrt[4]{a})^3;\ a^{\frac{1}{n}} = \sqrt[n]{a};\ 8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4.$

231. *To find the Meaning of a Negative Exponent.*

Let r denote any positive integer or fraction; then assuming law (I.), we have

$$a^r \cdot a^{-r} = a^{r-r} = a^0 = 1. \qquad \text{§ 77}$$

$$\therefore\ a^{-r} = 1 \div a^r.$$

That is, a^{-r} denotes the *reciprocal* of a^r, or it denotes that a^r is to be used as a *divisor.*

Hence a **Negative Exponent** denotes a power, or a power of a root, of its base, that is to be used as a *divisor.*

Thus $$a^{-3} = 1 \cdot a^{-3} = 1 \div a^3,$$

and $$ab^{-\frac{1}{2}}c^{-\frac{3}{5}} = a \div b^{\frac{1}{2}} \div c^{\frac{3}{5}}.$$

Hence the quality of an exponent may be changed, if the sign before the exponential expression be changed from × to ÷, or from ÷ to ×.

NOTE 1. The *arithmetical value* of an exponent denotes a power, or a power of a root, of its base; and its *quality* denotes whether this power is to be used as a *factor* or *divisor.*

NOTE 2. Fractional and negative exponents express no new ideas, and are not necessary to the notation of algebra; but they are very convenient, and greatly facilitate many operations. *Fractional* exponents simply afford another way of writing a power of a root; and *negative* exponents another way of writing a *divisor.*

232. Any *exponential factor* can be changed from the dividend to the divisor, or from the divisor to the dividend, provided the *quality* of its exponent be changed.

For this is simply multiplying or dividing both the dividend and divisor by the same exponential expression.

Thus $$\frac{a^{-2}}{c^2} = \frac{a^{-2} \cdot a^2c^{-2}}{c^2 \cdot a^2c^{-2}} = \frac{c^{-2}}{a^2}.$$

Also, $$\frac{a^{-1}x^2}{c^{-2}y^3} = \frac{c^2x^2}{ay^3},\ a^{-1}x^2c^2y^{-3}, \text{ or } \frac{1}{ax^{-2}c^{-2}y^3}.$$

Exercise 74.

Write with positive exponents

1. ax^{-4}. 2. $5a^{-3}x^{-5}$. 3. $4ax^{-7}y^{-1}$.

4. $\dfrac{1}{6a^{-2}x^{-\frac{1}{2}}}$. 8. $\dfrac{x^{-1}y^{-2}z^{-3}}{ab^{-1}c^{-2}}$.

5. $\dfrac{5a^{-3}x^{4}}{7y^{2}z^{-5}}$. 9. $\dfrac{b(x-y)^{-1}}{(x+y)^{-2}}$.

6. $\dfrac{5x^{-1}y^{-3}z^{2}}{7c^{-2}db^{-\frac{2}{3}}}$. 10. $\dfrac{a^{-m}b^{-n}}{x^{-c}y^{-s}z^{-r}}$.

7. $\dfrac{x^{-3}y^{-2}}{a^{-2}b^{-4}}$. 11. $\dfrac{(a+b)^{-1}x^{-7}y}{(a+b)^{-2}}$.

Find the value of

12. $8^{\frac{4}{3}}$. 14. $9^{-\frac{3}{2}}$. 16. $(\frac{27}{8})^{\frac{4}{3}}$. 18. $(-27)^{\frac{5}{3}}$.

13. $4^{-\frac{5}{2}}$. 15. $(\frac{4}{9})^{-\frac{1}{2}}$. 17. $(\frac{81}{16})^{-\frac{5}{4}}$. 19. $9^{-\frac{3}{2}}\cdot 8^{\frac{4}{3}}$.

Write with radical sign and positive integral exponents

20. $\dfrac{2}{x^{-\frac{4}{3}}}$. 21. $\dfrac{a^{-\frac{2}{3}}}{3}$. 22. $\dfrac{x^{-\frac{1}{a}}}{y^{-\frac{2}{c}}}$.

Simplify

23. $8^{-\frac{2}{3}}\cdot 4^{\frac{1}{2}}$. 24. $(\frac{1}{64})^{-\frac{2}{3}}\cdot(\frac{1}{9})^{\frac{1}{2}}$. 25. $(\frac{1}{25})^{-\frac{1}{2}}\cdot(27)^{-\frac{4}{3}}$.

233. *The* s*th root of the* q*th root of a number equals the* qs*th root of that number; that is,* $\sqrt[s]{\sqrt[q]{a}}=\sqrt[qs]{a}$.

For if a number be resolved into q equal factors and then each one of these q equal factors be resolved into will be resolved into *qs*

$= \sqrt[3]{\sqrt[4]{a}} = \sqrt[12]{a}.$

of exponents when m *and* n

any positive integers; then,

$= a^{\frac{p}{q}\ \frac{r}{s}};$ (I.)

$= a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} = a^{\frac{p}{q}\ \frac{r}{s}}.$ (II.)

$= (\sqrt[s]{\sqrt[q]{a^p}})^r$ § 230

$= (\sqrt[qs]{a^p})^r$ § 233

$= [(\sqrt[qs]{a})^p]^r$ § 227

$= (\sqrt[qs]{a})^{pr} = a^{\frac{pr}{qs}}.$ (III.)

$= \sqrt[s]{(ab)^r}$ § 230

$= \sqrt[s]{a^r b^r}$ § 108

$= \sqrt[s]{a^r} \times \sqrt[s]{b^r}$ § 124

$= a^{\frac{r}{s}} b^{\frac{r}{s}}.$ (IV.)

$= \sqrt[s]{(a \div b)^r}$ § 230

$= \sqrt[s]{a^r \div b^r}$ § 109

$= \sqrt[s]{a^r} \div \sqrt[s]{b^r}$ § 125

(V.)

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number does not change the value of the exponential expression.

For $$a^{\frac{r}{s}} = (a^{\frac{r}{s}})^{\frac{n}{n}} = a^{\frac{rn}{sn}}.$$

From this principle it follows that

$$a^{\frac{r}{s}+\frac{p}{q}} = a^{\frac{rq}{sq}+\frac{ps}{sq}} = a^{\frac{rq+ps}{sq}}.$$

236. *To prove the five laws of exponents when* m *and* n *denote negative numbers.*

Let h and k denote any positive integers or fractions; then by § 229, we have

$$a^{-h}a^{-k} = a^{-h-k}; \qquad \text{(I.)}$$

and $$a^{-h} \div a^{-k} = a^{-h} \cdot a^{k} = a^{-h+k}. \qquad \text{(II.)}$$

To prove (III.), $$(a^{-h})^{-k} = 1 \div \left(\frac{1}{a^h}\right)^k \qquad \text{§ 231}$$

$$= 1 \div \frac{1}{a^{hk}} \qquad \text{§ 109}$$

$$= a^{hk}. \qquad \text{(III.)}$$

To prove (IV.), $$(ab)^{-h} = \frac{1}{(ab)^h} \qquad \text{§ 231}$$

$$= \frac{1}{a^h b^h}$$

$$= \frac{1}{a^h} \times \frac{1}{b^h} = a^{-h}b^{-h}. \qquad \text{(IV.)}$$

To prove (V.), $$\left(\frac{a}{b}\right)^{-h} = 1 \div \left(\frac{a}{b}\right)^h$$

$$= 1 \div \frac{a^h}{b^h}$$

$$= \frac{b^h}{a^h} = \frac{a^{-h}}{b^{-h}}.$$

237. If we use the word *power* to signify what is indicated by any exponent, the five laws of exponents may be stated as follows:

(I.) *The product of the* m*th and the* n*th power of any base equals the* (m + n)*th power of that base.*

(II.) *The quotient of the* m*th power of any base divided by its* n*th power equals the* (m − n)*th power of that base.*

(III.) *The* n*th power of the* m*th power of any base equals the* mn*th power of that base.*

(IV.) *The* m*th power of the product of any number of factors equals the product of the* m*th powers of those factors.*

(V.) *The* m*th power of the quotient of any two numbers equals the quotient of their* m*th powers.*

We give the following illustrative examples:

(1) $$x^{\frac{3}{4}} \cdot x^{\frac{2}{3}} = x^{\frac{3}{4}+\frac{2}{3}} = x^{\frac{17}{12}} = \sqrt[12]{x^{17}}.$$

(2) $$a^{\frac{2}{3}} \cdot a^{\frac{1}{2}} \cdot a^{-\frac{7}{6}} = a^{\frac{2}{3}+\frac{1}{2}-\frac{7}{6}} = a^0 = 1.$$

(3) $$(x^{-\frac{2}{3}})^6 = x^{-\frac{2}{3}\times 6} = x^{-4} = 1 \div x^4.$$

(4) $$x^{-\frac{3}{2}} \div x^{-\frac{5}{6}} = x^{-\frac{3}{2}}x^{\frac{5}{6}} = x^{-\frac{3}{2}+\frac{5}{6}} = x^{-\frac{2}{3}} = 1 \div \sqrt[3]{x^2}.$$

(5) $$(4\,a^2b^{-2}c^{\frac{3}{5}})^{\frac{3}{2}} = 4^{\frac{3}{2}}(a^2)^{\frac{3}{2}}(b^{-2})^{\frac{3}{2}}(c^{\frac{3}{5}})^{\frac{3}{2}}$$
$$= 8\,a^3b^{-3}c^{\frac{9}{10}} = 8\,a^3\sqrt[10]{c^9} \div b^3.$$

(6) $$\left(\frac{4\,x^{-2}}{9\,y^3}\right)^{-\frac{3}{2}} = \frac{(4\,x^{-2})^{-\frac{3}{2}}}{(9\,y^3)^{-\frac{3}{2}}} = \frac{(9\,y^3)^{\frac{3}{2}}}{(4\,x^{-2})^{\frac{3}{2}}}$$
$$= \frac{27\,y^{\frac{9}{2}}}{8\,x^{-3}} = \frac{27\,x^3y^{\frac{9}{2}}}{8}.$$

(7) $$(9^{-\frac{3}{4}}x^{-6})^{-\frac{2}{3}} = (9^{-\frac{3}{4}})^{-\frac{2}{3}}(x^{-6})^{-\frac{2}{3}} = 9^{\frac{1}{2}}x^4 = 3\,x^4.$$

(8) $$\frac{a^{\frac{2}{3}}\sqrt{b^{-1}}}{b\sqrt[3]{a^{-2}}} \div \sqrt{\frac{a\sqrt{b^{-4}}}{b\sqrt{a^{-2}}}} = \frac{a^{\frac{2}{3}}b^{-\frac{1}{2}}}{ba^{-\frac{2}{3}}} \div \sqrt{\frac{ab^{-2}}{ba^{-1}}}.$$

$$= a^{\frac{4}{3}}b^{-\frac{3}{2}} \div (a^2b^{-3})^{\frac{1}{2}}$$

$$= a^{\frac{4}{3}}b^{-\frac{3}{2}} \div ab^{-\frac{3}{2}} = a^{\frac{1}{3}}.$$

Exercise 75.

Reduce to its simplest form

1. $a^6 \cdot a^5 \cdot a^{-7}$.
2. $a^{-5} \cdot a^{-2}a^8$.
3. $7a^{-\frac{3}{4}} \cdot 3a^{\frac{5}{4}}$.
4. $\sqrt[2a]{x} \cdot \sqrt[a]{x^2}$.
5. $2x^{-\frac{5}{4}} \cdot x^{-\frac{3}{4}}$.
6. $x^8 \div x^{-2}$.
7. $x^{-4} \div x^{-5}$.
8. $x^{\frac{3}{2}} \div x^{-\frac{3}{5}}$.
9. $y^{-\frac{4}{5}} \div y^{-\frac{7}{5}}$.
10. $(4x^2b^{-3})^{\frac{3}{2}}$.
11. $(8y^{-3}z^{-6})^{-\frac{2}{3}}$.
12. $(9x^{-4}y^{-\frac{3}{4}})^{-\frac{3}{2}}$.
13. $a^{\frac{3}{2}}x^{\frac{1}{2}}y^{\frac{3}{4}} \cdot a^{-\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{5}{4}}$.
14. $a^{-\frac{1}{2}}x^{-\frac{3}{4}}y^{-\frac{3}{2}} \cdot a^3x^{\frac{7}{8}}y^{\frac{9}{4}}$.
15. $3x^{-\frac{7}{8}}y^{\frac{3}{2}} \cdot 5x^{-\frac{1}{2}}y^{-\frac{1}{2}}$.
16. $14x^{\frac{3}{2}}y^{\frac{2}{3}} \div (7x^{-\frac{1}{2}}y^{-\frac{1}{3}})$.
17. $\sqrt[5]{a^{-3}} \div \sqrt[5]{a^{-8}}$.
18. $\sqrt[3]{x^{-2}y^{-1}} \div \sqrt[3]{x^{-8}y^{-4}}$.
19. $\sqrt[3]{x^m} \cdot \sqrt[4]{x^m} \div \sqrt[12]{x^m}$.
20. $\sqrt[6]{a^{4b}x^6} \cdot (a^{\frac{2}{3}}x^{-1})^{-b}$.
21. $\sqrt[3]{ab^{-1}c^{-2}} \cdot (a^{-1}b^{-2}c^{-4})^{-\frac{1}{6}}$.
22. $\sqrt[5]{a^2b^{10}c^5} \div \sqrt[3]{ab^6c^3}$.
23. $\sqrt{a^3b^{-2}} \div \sqrt[3]{a^{-4}b^5}$.
24. $\sqrt[4]{(a+b)^6} \cdot (a^2-b^2)^{-\frac{1}{2}}$.
25. $\sqrt[3]{x^{-1}\sqrt{y^3}} \div \sqrt{y\sqrt[3]{x}}$.
26. $\sqrt[n]{a^{n+k}b^{2n-k}} \div (a^{\frac{1}{n}}b^{-\frac{1}{n}})^k$.
27. $\sqrt[3]{(a+b)^5} \cdot (a+b)^{-\frac{2}{3}}$.
28. $\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^{-3} \div \left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5$.

29. $\left\{\frac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}}\cdot\left(\frac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2\div\frac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right\}^6$ 31. $\left(\frac{a^{-3}}{b^{-\frac{2}{3}}}\right)^{-\frac{3}{2}}\div\left(\frac{a^{-\frac{1}{4}}b^{\frac{1}{2}}}{a^2c^{-1}}\right)^{-4}.$

30. $\left(\frac{x^{-2}y^3}{x^3y^{-2}}\right)^{-\frac{1}{5}}\cdot\left(\frac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1}.$ 32. $\left(\frac{a^{-\frac{2}{3}}x^{\frac{1}{2}}}{x^{-1}a}\right)^2\div\sqrt[3]{\frac{a^{-1}}{x^{-3}}}.$

238. The examples in the last Exercise illustrate the advantage of fractional and negative exponents in the multiplication, division, involution, and evolution of monomials that involve fractional and negative powers. Their still greater advantage in the case of polynomials will be illustrated in the following examples.

EXAMPLE 1. Multiply $\sqrt[3]{a}+1+1\div\sqrt[3]{a}$ by $\sqrt[3]{a}+1\div\sqrt[3]{a}-1$.

The terms $+1$ and -1 may be regarded as the coefficients of a^0; hence, arranging both expressions according to the descending powers of a, we have

$$\begin{array}{lllll}
a^{\frac{1}{3}} & +1 & +a^{-\frac{1}{3}} & & \\
a^{\frac{1}{3}} & -1 & +a^{-\frac{1}{3}} & & \\
\hline
a^{\frac{2}{3}} & +a^{\frac{1}{3}} & +1 & & \\
 & -a^{\frac{1}{3}} & -1 & -a^{-\frac{1}{3}} & \\
 & & +1 & +a^{-\frac{1}{3}} & +a^{-\frac{2}{3}} \\
\hline
a^{\frac{2}{3}} & & +1 & & +a^{-\frac{2}{3}}
\end{array}$$

EXAMPLE 2. Divide $16\,a^{-3}+5\,a^{-1}-6\,a^{-2}+6$ by $1+2\,a^{-1}$.

Arrange in ascending powers of a.

$$\begin{array}{llll|l}
16\,a^{-3} & -\ 6\,a^{-2} & +5\,a^{-1} & +6 & 2\,a^{-1}+1 \\
\cline{5-5}
16\,a^{-3} & +\ 8\,a^{-2} & & & 8\,a^{-2}-7\,a^{-1}+6 \\
\cline{1-3}
 & -14\,a^{-2} & +5\,a^{-1} & & \\
 & -14\,a^{-2} & -7\,a^{-1} & & \\
\cline{2-4}
 & & 12\,a^{-1} & +6 & \\
 & & 12\,a^{-1} & +6 & \\
\cline{3-4}
\end{array}$$

Example 3. Find the square root of

$$4x + 2x^{\frac{7}{6}} - 4x^{\frac{5}{6}} - 4x^{\frac{4}{3}} + x^{\frac{5}{3}} + x^{\frac{2}{3}}.$$

Arrange in descending powers of x.

$$\begin{array}{r|l}
 & x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x^{\frac{7}{6}} + 4x - 4x^{\frac{5}{6}} + x^{\frac{2}{3}} \;\underline{|\, x^{\frac{5}{6}} - 2x^{\frac{3}{6}} + x^{\frac{1}{3}}} \\
 & \underline{x^{\frac{5}{3}} \qquad\qquad\qquad\qquad\qquad} \\
2x^{\frac{5}{6}} - 2x^{\frac{3}{6}} & - 4x^{\frac{4}{3}} + 2x^{\frac{7}{6}} + 4x \\
 & \underline{- 4x^{\frac{4}{3}} \qquad\qquad + 4x \qquad\qquad\qquad} \\
2x^{\frac{5}{6}} - 4x^{\frac{3}{6}} + x^{\frac{1}{3}} & 2x^{\frac{7}{6}} \qquad\qquad - 4x^{\frac{5}{6}} + x^{\frac{2}{3}} \\
 & \underline{2x^{\frac{7}{6}} \qquad\qquad - 4x^{\frac{5}{6}} + x^{\frac{2}{3}}}
\end{array}$$

Exercise 76.

Multiply

1. $a^{\frac{4}{3}} + a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$ by $a^{\frac{2}{3}} - b^{\frac{2}{3}}$.
2. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.
3. $3x^{\frac{1}{3}} - 5 + 8x^{-\frac{1}{3}}$ by $4x^{\frac{1}{3}} + 3x^{-\frac{1}{3}}$.
4. $3a^{\frac{3}{5}} - 4a^{\frac{1}{5}} - a^{-\frac{1}{5}}$ by $3a^{\frac{1}{5}} + a^{-\frac{1}{5}} - 6a^{-\frac{3}{5}}$.
5. $a^{\frac{4}{3}} + a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$ by $a^{\frac{4}{3}} - a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.
6. $x^{\frac{7}{4}} - x^{\frac{3}{2}} + x^{\frac{5}{4}} - x$ by $x^{\frac{3}{4}} + x^{\frac{1}{2}}$.
7. $x^{\frac{5}{4}} - x^{\frac{3}{4}} + x^{\frac{1}{4}} - x^{-\frac{1}{4}}$ by $x^{\frac{3}{4}} + x^{\frac{1}{4}}$.
8. $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}} - c^{\frac{1}{3}}a^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$.
9. $x^{n} + x^{\frac{n}{2}} + 1$ by $x^{-n} + x^{-\frac{n}{2}} + 1$.
10. $\frac{1}{8}a^{\frac{3}{2}} - \frac{1}{12}ab^{\frac{1}{2}} + \frac{1}{18}a^{\frac{1}{2}}b - \frac{1}{27}b^{\frac{3}{2}}$ by $\frac{1}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{2}}$.
11. $c^{x} + 2c^{-x} - 7$ by $5 - 3c^{-x} + 2c^{x}$.

Divide

12. $21x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$ by $3x^{\frac{1}{3}} + 1$.

13. $15a - 3a^{\frac{1}{3}} - 2a^{-\frac{1}{3}} + 8a^{-1}$ by $5a^{\frac{2}{3}} + 4$.

14. $5b^{\frac{2}{3}} - 6b^{\frac{1}{3}} - 4b^{-\frac{2}{3}} - 4b^{-\frac{1}{3}} - 5$ by $b^{\frac{1}{6}} - 2b^{-\frac{1}{6}}$.

15. $21a^{3x} + 20 - 27a^{x} - 26a^{2x}$ by $3a^{x} - 5$.

16. $x^{\frac{3}{4}}y^{-\frac{3}{4}} + 2 + x^{-\frac{3}{4}}y^{\frac{3}{4}}$ by $x^{\frac{1}{4}}y^{-\frac{1}{4}} - 1 + x^{-\frac{1}{4}}y^{\frac{1}{4}}$.

17. $a^{\frac{5}{2}} + a^{2}b^{\frac{1}{3}} - a^{\frac{3}{2}}b^{\frac{2}{3}} - ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{3}}$.

18. $x^{\frac{7}{3}}y^{-\frac{14}{5}} + y^{\frac{14}{5}}x^{-\frac{7}{3}}$ by $x^{\frac{1}{3}}y^{-\frac{2}{5}} + y^{\frac{2}{5}}x^{-\frac{1}{3}}$.

19. $a^{\frac{4}{3}} - 2 + a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}} - a^{-\frac{2}{3}}$.

20. $8c^{-n} - 8c^{n} + 5c^{3n} - 3c^{-3n}$ by $5c^{n} - 3c^{-n}$.

Find the square root of

21. $25a^{\frac{4}{3}} + 16 - 30a - 24a^{\frac{1}{3}} + 49a^{\frac{2}{3}}$.

22. $9x - 12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1}$.

23. $4x^{2}a^{-2} + 12xa^{-1} + 25 + 24x^{-1}a + 16x^{-2}a^{2}$.

24. $25x^{2}y^{-2} + \frac{1}{4}y^{2}x^{-2} - 20xy^{-1} - 2yx^{-1} + 9$.

25. $x^{\frac{8}{5}} - 2a^{-\frac{3}{5}}x^{\frac{11}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{\frac{1}{5}}x^{\frac{7}{5}} + a^{\frac{8}{5}}$.

26. $4x^{n} + 9x^{-n} + 28 - 24x^{-\frac{1}{2}n} - 16x^{\frac{1}{2}n}$.

27. $9x^{-4} - 18x^{-3}\sqrt{y} + 15y \div x^{2} - 6\sqrt{y^{3}} \div x + y^{2}$.

239. The following examples will illustrate the application of the Binomial Theorem and the principles of Chapter X to binomials whose terms involve fractional and negative exponents.

(1) $$(2a^{\frac{1}{2}} - x^{-\frac{1}{2}})^3 = (2a^{\frac{1}{2}})^3 + 3(2a^{\frac{1}{2}})^2(-x^{-\frac{1}{2}})$$
$$+3(2a^{\frac{1}{2}})(-x^{-\frac{1}{2}})^2 + (-a^{-\frac{1}{2}})^3$$
$$= 8a^{\frac{3}{2}} - 12ax^{-\frac{1}{2}} + 6a^{\frac{1}{2}}x^{-1} - a^{-\frac{3}{2}}.$$

(2) $$(x^{\frac{3}{2}} + y^{\frac{3}{2}})(x^{\frac{3}{2}} - y^{\frac{3}{2}}) = (x^{\frac{3}{2}})^2 - (y^{\frac{3}{2}})^2 = x^3 - y^3.$$ § 140

(3) $(7x - 9y^{-1})(7x + 9y^{-1}) = 49x^2 - 81y^{-2}.$

(4) $$(4x - 5x^{-1})(4x + 3x^{-1}) = 16x^2 + (3x^{-1} - 5x^{-1})4x$$
$$- 15x^{-2}$$ § 141
$$= 16x^2 - 8 - 15x^{-2}.$$

(5) $$(x^{\frac{5}{3}} - 1) \div (x^{\frac{1}{3}} - 1) = [(x^{\frac{1}{3}})^5 - 1^5] \div (x^{\frac{1}{3}} - 1)$$
$$= (x^{\frac{1}{3}})^4 + (x^{\frac{1}{3}})^3 + (x^{\frac{1}{3}})^2 + x^{\frac{1}{3}} + 1$$ § 142
$$= x^{\frac{4}{3}} + x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1.$$

(6) $$(x^{\frac{3}{2}} + 27) \div (x^{\frac{1}{2}} + 3) = [(x^{\frac{1}{2}})^3 + 3^3] \div (x^{\frac{1}{2}} + 3)$$
$$= x - 3x^{\frac{1}{2}} + 9.$$ § 144

Exercise 77.

Write down the value of

1. $(a^{-\frac{3}{4}} + b^{\frac{2}{3}})^2.$ **3.** $(m^{\frac{1}{2}} + n^{\frac{2}{3}})^3.$ **5.** $(r^{-\frac{1}{3}} - s^{-\frac{1}{2}})^3.$

2. $(x^{\frac{4}{5}} - y^{-\frac{2}{3}})^2.$ **4.** $(c^2 - b^{\frac{2}{3}})^3.$ **6.** $(2a^{\frac{1}{2}} - a^{-\frac{1}{2}})^3.$

7. $(r^{-2} + b^{\frac{1}{2}})^4.$ **11.** $(a^{\frac{2}{3}} + b^{-\frac{3}{4}})^5.$

8. $(r^2 - 3n^{-\frac{1}{2}})^4.$ **12.** $(x^{\frac{1}{2}} - y^{-\frac{2}{3}})^6.$

9. $(\frac{1}{2}\sqrt{a} - \frac{1}{3}\sqrt[3]{b})^4.$ **13.** $(x^{\frac{1}{4}} + 1)(x^{\frac{1}{4}} - 1).$

10. $(a^{-\frac{3}{5}} - 2b^2c^{\frac{1}{3}})^4.$ **14.** $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}}).$

15. $(4x^{\frac{3}{2}} + 3a^{-\frac{1}{2}})(4x^{\frac{3}{2}} - 3a^{-\frac{1}{2}}).$

16. $(3x - 5a^{-1})(3x + 2a^{-1}).$

17. $(ab - c^{\frac{1}{2}})(ab + 5x^{-1})$.

18. $(x - 9a) \div (x^{\frac{1}{2}} + 3a^{\frac{1}{2}})$.

19. $(a^{-2x} - 16) \div (a^{-x} - 4)$.

20. $(x^{-3a} + 8) \div (x^{-a} + 2)$.

21. $(c^{2x} - c^{-x}) \div (c^{x} - c^{-\frac{1}{2}x})$.

22. $(1 - 8a^{-3}) \div (1 - 2a^{-1})$.

23. $(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)(x^{\frac{1}{3}} - 1)$.

24. $(x^{-4} - 1) \div (x^{-1} + 1)$.

25. $(x^{5n} + 32) \div (x^{n} + 2)$.

26. $(x^{3} - y^{3}) \div (x^{\frac{3}{2}} - y^{\frac{3}{2}})$.

27. $(x^{2} + y^{2}) \div (x^{\frac{2}{3}} + y^{\frac{2}{3}})$.

28. $(x - 243y^{\frac{5}{3}}) \div (x^{\frac{1}{5}} - 3y^{\frac{1}{3}})$.

240. By the laws of the Binomial Theorem, we have

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{1 \cdot 2}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}a^{m-3}x^3 + \cdots \qquad \text{(F)}$$

which is the Binomial Theorem expressed in symbols.

241. The *n*th Term. From (F) of § 240 it follows that the *n*th term in the expansion of $(a+x)^m$ is

$$\frac{m(m-1)\cdots \text{to } n-1 \text{ factors}}{1 \cdot 2 \cdot 3 \cdots \text{to } n-1 \text{ factors}}a^{m-(n-1)}x^{n-1}.$$

EXAMPLE. Find the seventh term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.

Here $a = \frac{4x}{5}$, $x = -\frac{5}{2x}$, $m = 9$, $n = 7$, $n - 1 = 6$.

Substituting these values in the formula, we have

$$\text{the seventh term} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\left(\frac{4x}{5}\right)^3\left(\frac{-5}{2x}\right)^6$$
$$= 10500\,x^{-3}.$$

242. Binomial with Any Exponent. The Binomial Theorem, or formula (F), holds true also when the exponent m is fractional or negative, provided that a, the first term of the binomial, is *numerically greater* than x, the second term. In this case, however, the expansion has an infinite number of terms; that is, it never ends.

NOTE. The proof of the Binomial Theorem for any exponent will be found in the "College Algebra," where it is proved as a particular case of a more general formula.

EXAMPLE 1. Expand $(c-d^2)^{\frac{3}{5}}$.

Applying the *laws* of the Binomial Theorem, we have

$$(c-d^2)^{\frac{3}{5}} = c^{\frac{3}{5}} + \tfrac{3}{5}c^{-\frac{2}{5}}(-d^2) - \tfrac{3}{25}c^{-\frac{7}{5}}(-d^2)^2$$
$$+ \tfrac{7}{125}c^{-\frac{12}{5}}(-d^2)^3 - \cdots \qquad (1)$$
$$= c^{\frac{3}{5}} - \tfrac{3}{5}c^{-\frac{2}{5}}d^2 - \tfrac{3}{25}c^{-\frac{7}{5}}d^4 - \tfrac{7}{125}c^{-\frac{12}{5}}d^6 - \cdots. \qquad (2)$$

The coefficient of the third term is $\frac{3}{5}\left(-\frac{2}{5}\right)\div 2$, or $-\frac{3}{25}$; that of the fourth term is $\left(-\frac{3}{25}\right)\left(-\frac{7}{5}\right)\div 3$, or $\frac{7}{125}$; and so on.

In performing the operations indicated in (1), first note the number of negative numerical factors in each term to determine the quality of its numerical coefficient: thus in the third term there are three negative numerical factors, $-\frac{3}{25}$ and $(-1)^2$.

Equality (1) or (2) is true only when $c > d^2$ *numerically.*

EXAMPLE 2. Expand $\dfrac{1}{\sqrt[4]{a^2-c^{\frac{1}{2}}}}$, or $(a^2-c^{\frac{1}{2}})^{-\frac{1}{4}}$.

$$(a^2-c^{\frac{1}{2}})^{-\frac{1}{4}} = (a^2)^{-\frac{1}{4}} - \tfrac{1}{4}(a^2)^{-\frac{5}{4}}(-c^{\frac{1}{2}}) + \tfrac{5}{32}(a^2)^{-\frac{9}{4}}(-c^{\frac{1}{2}})^2$$
$$- \tfrac{15}{128}(a^2)^{-\frac{13}{4}}(-c^{\frac{1}{2}})^3 + \cdots$$
$$= a^{-\frac{1}{2}} + \tfrac{1}{4}a^{-\frac{5}{2}}c^{\frac{1}{2}} + \tfrac{5}{32}a^{-\frac{9}{2}}c + \tfrac{15}{128}a^{-\frac{13}{2}}c^{\frac{3}{2}} + \cdots. \qquad (1)$$

Equality (1) is true only when $a^2 > c^{\frac{1}{2}}$ numerically.

Example 3. Find the cube root of 127.

$$127 = 125 + 2 = 5^3 + 2 = 5^3\left(1 + \frac{2}{5^3}\right).$$

$$\therefore\ \sqrt[3]{127} = 5\left(1 + \frac{2}{5^3}\right)^{\frac{1}{3}}$$

$$= 5\left(1 + \frac{2}{3 \cdot 5^3} - \frac{4}{9 \cdot 5^6} + \frac{40}{81 \cdot 5^9} - \cdots\right)$$

$$= 5\,(1 + 0.0053333 - 0.0000284 + 0.0000002 - \cdots)$$

$$= 5\,(1.0053051) = 5.0265255 -.$$

Here we put $127 = 125 + 2$, because 125 is the perfect cube that is nearest in value to 127. To find $\sqrt[3]{120}$, we would put $120 = 125 - 5$. The smaller the second term of the binomial, the more rapidly the successive terms of the expansion decrease.

Example 4. Find the sixth term of $(b^2 - c^2x^2)^{-\frac{1}{2}}$.

Here $a = b^2$, $x = -c^2x^2$, $m = -\frac{1}{2}$, $n = 6$, $n - 1 = 5$.

Substituting these values in the formula of § 241, we have

$$\text{the sixth term} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(-\frac{9}{2})}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(b^2)^{-\frac{1}{2}-5}(-c^2x^2)^5 \qquad (1)$$

$$= \frac{7 \cdot 9}{2 \cdot 4 \cdot 2^5}\,b^{-11}c^{10}x^{10} = \frac{63}{256}\,b^{-11}c^{10}x^{10}. \qquad (2)$$

There are 10 negative numerical factors in (1); hence the numerical coefficient of the term in (2) is positive.

Exercise 78.

Expand to four terms:

1. $(1 + x)^{-1}$.
2. $(1 + x)^{-2}$.
3. $(1 + x)^{-3}$.
4. $(1 - x)^{-1}$.
5. $(1 - x)^{-\frac{1}{2}}$.
6. $(1 - x)^{-2}$.
7. $(1 - x)^{-4}$.
8. $(1 + 2x)^{-4}$.
9. $(2 - x)^{-3}$.
10. $(1 - 3x)^{-\frac{2}{3}}$.
11. $(1 - 5x)^{\frac{3}{5}}$.
12. $(b^{\frac{1}{2}} - c^{-\frac{3}{2}})^{-\frac{3}{4}}$.
13. $\dfrac{1}{\sqrt{x^2 - y^3}}$.
14. $\dfrac{1}{a^{\frac{2}{3}} - b^{-\frac{1}{2}}}$.

Find the

15. Fifth term of $(2x - x^2)^{12}$.

16. Sixth term of $(2a - \frac{1}{2})^8$.

17. Fourth term and ninth term of $(3x - y)^{11}$.

18. Fifth term and eighth term of $(a - 2\sqrt{b})^{20}$.

19. Sixth term and eleventh term of $(\frac{1}{2}a - b\sqrt{b})^{22}$.

20. Fifth term of $(1 - a^2)^{-\frac{3}{2}}$.

21. Seventh term of $(x^{-1} - y^{\frac{1}{2}})^{\frac{4}{5}}$.

22. Third term and eleventh term of $(1 + 2x)^{\frac{11}{2}}$.

23. Fifth term of $(c^{-2} + e^{-\frac{1}{2}})^{-4}$.

24. Sixth term of $(x^{-\frac{2}{3}} - a^2b^{\frac{3}{4}})^{-\frac{3}{5}}$.

Find to four places of decimals the value of

25. $\sqrt[3]{31}$. **27.** $\sqrt[5]{29}$. **29.** $\sqrt[4]{620}$. **31.** $\sqrt[3]{998}$.

26. $\sqrt[4]{17}$. **28.** $\sqrt[3]{122}$. **30.** $\sqrt[5]{31}$. **32.** $\sqrt[5]{3128}$.

Expand to four terms:

33. $(8 + 12a)^{\frac{2}{3}}$.

34. $(1 - 3x)^{\frac{1}{3}}$.

35. $(1 - 3x)^{-\frac{1}{3}}$.

36. $(a^2 + c^{\frac{1}{2}})^{\frac{3}{4}}$.

37. $(c - d^2)^{-\frac{3}{5}}$.

38. $(9 + 2x)^{\frac{1}{2}}$.

39. $(4a - 8x)^{-\frac{1}{2}}$.

40. $(c^2a^{-\frac{1}{2}} - b^2e^{-\frac{2}{3}})^{-\frac{3}{5}}$.

41. $\dfrac{a}{(cb^{-\frac{2}{3}} - x^2y^{-\frac{1}{2}})^{\frac{3}{4}}}$.

CHAPTER XX.

SURD AND IMAGINARY NUMBERS.

243. Any rational expression can be written in the form of a surd. (For the definition of a surd, see § 127.)

Thus $3 = \sqrt{9},\ \sqrt[3]{27},\ \sqrt[4]{81}$, or $\sqrt[5]{243}$;

and $a^2 = \sqrt{a^4},\ \sqrt[3]{a^6},\ \sqrt[4]{a^8}$, or $\sqrt[5]{a^{10}}$.

244. The **Order** of a surd is indicated by the index of the root. Thus $\sqrt{3}$ and $5^{\frac{3}{2}}$ are surds of the *second order*, or **Quadratic** surds; also $\sqrt[3]{4}$ and $a^{\frac{2}{3}}$ are surds of the *third order*, or **Cubic** surds; and $\sqrt[n]{a}$ and $7^{\frac{m}{n}}$ are surds of the nth *order*.

245. A surd is in its *simplest form* when the expression under the radical sign is *integral* and as *small*, or of as *low* a *degree*, as possible.

Thus the simplest form of the surd $\sqrt{8}$ is $2\sqrt{2}$, and the simplest form of $\sqrt[3]{x^4y^5}$ is $xy\sqrt[3]{xy^2}$.

246. Surds are reduced to their simplest form by the law of § 124, or $\sqrt[n]{a^nb} = a \cdot \sqrt[n]{b}$. (1)

The cases that most frequently occur are the two following:

I. When the expression under the radical sign is integral, and can be resolved into two factors, one of which is a perfect power of the same degree as the radical.

In this case, by law (1) we have

(1) $$\sqrt{x^3y^2} = \sqrt{(xy)^2 \cdot x} = xy\sqrt{x}.$$

(2) $$\sqrt[3]{135} = \sqrt[3]{3^3 \cdot 5} = 3\sqrt[3]{5}.$$

(3) $$7\sqrt{50\,x^4y^3} = 7\sqrt{(5\,x^2y)^2 \cdot 2\,y}$$
$$= 7 \times 5\,x^2y\sqrt{2\,y} = 35\,x^2y\sqrt{2\,y}.$$

(4) $$5\sqrt[3]{128\,x^5y^4} = 5\sqrt[3]{(4\,xy)^3 \cdot 2\,x^2y}$$
$$= 5 \times 4\,xy\sqrt[3]{2\,x^2y} = 20\,xy\sqrt[3]{2\,x^2y}.$$

II. When the expression under the radical sign is a fraction, we multiply both its numerator and denominator by such a number as will make the denominator a perfect power of the same degree as the radical.

In this case, by law (1) we have

(1) $$2\sqrt{\frac{3}{10}} = 2\sqrt{\frac{30}{10^2}} = 2\sqrt{\left(\frac{1}{10}\right)^2 \times 30}$$
$$= 2 \times \tfrac{1}{10}\sqrt{30} = \tfrac{1}{5}\sqrt{30}.$$

(2) $$3\sqrt{\frac{5}{12}} = 3\sqrt{\frac{15}{36}} = 3\sqrt{\left(\frac{1}{6}\right)^2 \times 15}$$
$$= 3 \times \tfrac{1}{6}\sqrt{15} = \tfrac{1}{2}\sqrt{15}.$$

(3) $$\sqrt[3]{\frac{5}{49}} = \sqrt[3]{\frac{35}{7^3}} = \sqrt[3]{\left(\frac{1}{7}\right)^3 \times 35} = \tfrac{1}{7}\sqrt[3]{35}.$$

Exercise 79.

Reduce to the simplest form

1. $\sqrt{147}$. 4. $2\sqrt{720}$. 7. $5\sqrt{245}$.

2. $\sqrt{288}$. 5. $\sqrt[3]{256}$. 8. $\sqrt[3]{1029}$.

3. $3\sqrt{150}$. 6. $\sqrt[3]{432}$. 9. $\sqrt[4]{3125}$.

10. $\sqrt[3]{-2187}$.

11. $\sqrt{27a^3b^5}$.

12. $\sqrt[3]{-108x^4y^3}$.

13. $\sqrt[n]{x^{3n}y^{2n+5}}$.

14. $\sqrt[p]{x^{a+p}y^{2p}}$.

15. $\sqrt{a^3+2a^2b+ab^2}$.

16. $\frac{1}{2}\sqrt{\frac{1}{3}}$.

17. $\frac{2}{3}\sqrt{\frac{4}{5}}$.

18. $\frac{4}{11}\sqrt{\frac{77}{2}}$.

19. $3\sqrt[3]{\frac{2}{3}}$.

20. $\frac{3xy}{z}\sqrt{\frac{5z^2}{9x^2y}}$.

21. $\frac{3x}{2a}\sqrt[3]{\frac{27a^4}{x^4}}$.

22. $\frac{2b}{a}\sqrt[4]{\frac{a^4}{8b^3}}$.

23. $a\sqrt[5]{\frac{b^2}{a^3}}$.

24. $\frac{a}{x}\sqrt[n]{\frac{x^{n+1}}{a^{n-2}}}$.

25. $\frac{a}{b^n}\sqrt{\frac{b^{2n+1}}{a^5}}$.

26. $(a+b)\sqrt{\frac{a-b}{a+b}}$.

27. $\frac{c}{b}\sqrt{\frac{a^2-x^2}{x^3c^3}}$.

247. Surds which when reduced to the simplest form have the same surd factor are said to be **like** or **similar.** Other surds are said to be **unlike.**

Thus $2\sqrt{3}$ and $\frac{1}{2}\sqrt{3}$ are *like* surds, so also are $3\sqrt{a}$ and $\sqrt{4a}$, for $\sqrt{4a}\equiv 2\sqrt{a}$.

248. To *add or subtract* surds, we reduce them to their simplest form, and unite those that are similar.

EXAMPLE 1. Simplify $\sqrt[3]{135}+\sqrt[3]{40}$.

$$\begin{aligned} \sqrt[3]{135} &= \sqrt[3]{3^3\times 5} = 3\sqrt[3]{5} \\ \sqrt[3]{40} &= \sqrt[3]{2^3\times 5} = 2\sqrt[3]{5} \\ \therefore \text{Sum} &= 5\sqrt[3]{5} \end{aligned}$$

EXAMPLE 2. Simplify $4\sqrt{128}+4\sqrt{75}-5\sqrt{162}$.

$$\begin{aligned} 4\sqrt{128} &= 4\sqrt{8^2\times 2} = 32\sqrt{2} \\ 4\sqrt{75} &= 4\sqrt{5^2\times 3} = 20\sqrt{3} \\ -5\sqrt{162} &= -5\sqrt{9^2\times 2} = -45\sqrt{2} \\ \therefore \text{Sum} &= 20\sqrt{3}-13\sqrt{2}. \end{aligned}$$

Exercise 80.

Find the value of

1. $\sqrt{27}+\sqrt{48}$.
2. $2\sqrt{180}-\sqrt{405}$.
3. $2\sqrt{28}-\sqrt{63}$.
4. $5\sqrt{208}-3\sqrt{325}$.
5. $\sqrt{512}-\sqrt{50}-\sqrt{98}$.
6. $3\sqrt{12}-\sqrt{27}+2\sqrt{75}$.
7. $\sqrt{44}-5\sqrt{176}+2\sqrt{99}$.
8. $2\sqrt{363}-5\sqrt{243}+\sqrt{192}$.
9. $2\sqrt[3]{189}+3\sqrt[3]{875}-7\sqrt[3]{56}$.
10. $\sqrt[3]{81}-7\sqrt[3]{192}+4\sqrt[3]{648}$.
11. $\sqrt{252}-\sqrt{294}-48\sqrt{\frac{1}{6}}$.
12. $4\sqrt{63}+5\sqrt{7}-8\sqrt{28}$.
13. $\sqrt[3]{a^2}+\frac{1}{2}\sqrt[3]{a^2}-3\sqrt[3]{27a^2}$.
14. $\sqrt[3]{54}+\sqrt{\frac{1}{2}}-\frac{3}{4}\sqrt{\frac{2}{9}}$.
15. $\sqrt[3]{27c^4}-\sqrt[3]{8c^4}+\sqrt[3]{125c}$.
16. $\sqrt[5]{a^5b}-\sqrt[5]{b^6}+\sqrt[5]{32b}$.
17. $\sqrt{a^4x}+\sqrt{b^4x}-\sqrt{4a^2b^2x}$.
18. $3\sqrt{147}-\frac{7}{3}\sqrt{\frac{1}{3}}-\sqrt{\frac{1}{27}}$.
19. $3\sqrt{\frac{4}{5}}+3\sqrt{\frac{9}{5}}-\sqrt[4]{\frac{25}{81}}$.
20. $\frac{1}{18}\sqrt[3]{72}-\frac{1}{3}\sqrt[3]{\frac{1}{3}}+6\sqrt[3]{21\frac{1}{3}}$.

249. A surd of any order may be reduced to an equal surd of a different order.

For by § 235 the value of a surd is not changed by multiplying or dividing the index of the root and the exponent of the power by the same number.

Thus $\sqrt{3} = \sqrt[8]{3^4}$; for $3^{\frac{1}{2}} = 3^{\frac{4}{8}}$, § 235

$\sqrt[n]{a} = \sqrt[nr]{a^r}$; for $a^{\frac{1}{n}} = a^{\frac{r}{nr}}$,

and $\sqrt[6]{27\,a^3b^6} = \sqrt[6]{(3\,ab^2)^3} = \sqrt{3\,ab^2}$; for $(3\,ab^2)^{\frac{3}{6}} = (3\,ab^2)^{\frac{1}{2}}$.

250. Surds of different orders may be reduced to surds of the same order. This order may be *any* common multiple of each of the given orders, but it is usually most convenient to choose the least common multiple.

EXAMPLE 1. Reduce $\sqrt[3]{a^2}$, $\sqrt[4]{b^3}$, $\sqrt[6]{c^5}$ to surds of the same order.

The L. C. M. of 3, 4, 6 is 12; reducing the given surds to surds of the twelfth order, we obtain $\sqrt[12]{a^8}$, $\sqrt[12]{b^9}$, $\sqrt[12]{c^{10}}$.

EXAMPLE 2. Which is the greater $\sqrt[3]{6}$ or $\sqrt[4]{10}$?

Reducing the given surds to surds of the same order, we have

$$\sqrt[3]{6} = \sqrt[12]{6^4} = \sqrt[12]{1296}, \quad (1)$$

and
$$\sqrt[4]{10} = \sqrt[12]{10^3} = \sqrt[12]{1000}. \quad (2)$$

From their values in (1) and (2), it follows that $\sqrt[3]{6} > \sqrt[4]{10}$.

251. The **product** of two or more surds of the *same order* can be readily found from the law

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}. \quad \text{§ 124}$$

EXAMPLE 1. Multiply $\sqrt{7}$ by $\sqrt{28}$.

$$\sqrt{7} \times \sqrt{28} = \sqrt{7} \times 2\sqrt{7} = 2 \times 7 = 14.$$

EXAMPLE 2. Multiply $2\sqrt{14} \times \sqrt{21}$.

$$2\sqrt{14} \times \sqrt{21} = 2\sqrt{7 \times 2 \times 7 \times 3} = 14\sqrt{6}.$$

When the surds are of different orders, they should be reduced to equivalent surds of the same order.

Example 3. Multiply $\sqrt{3}$ by $\sqrt[3]{2}$.

$$\sqrt{3} \times \sqrt[3]{2} = \sqrt[6]{3^3} \times \sqrt[6]{2^2} = \sqrt[6]{3^3 \times 2^2} = \sqrt[6]{108}.$$

Example 4. Multiply $2\sqrt{3} + 3\sqrt{2}$ by $4\sqrt{3} - 5\sqrt{2}$.

The work may be arranged as below:

$$\begin{array}{l}
\quad 2\sqrt{3} + \ \ 3\sqrt{2} \\
\quad 4\sqrt{3} - \ \ 5\sqrt{2} \\
\hline
8 \times 3 + 12\sqrt{6} \\
\qquad\quad - 10\sqrt{6} - 15 \times 2 \\
\hline
\quad 24 + \ \ 2\sqrt{6} - 30 = 2\sqrt{6} - 6
\end{array}$$

The coefficient of a surd may be brought under the radical sign by reducing it to the form of a surd of the same order.

Thus $\qquad 5\sqrt{3} = \sqrt{25} \times \sqrt{3} = \sqrt{75},$

and $\qquad x\sqrt[5]{x^3} = \sqrt[5]{x^5} \times \sqrt[5]{x^3} = \sqrt[5]{x^8}.$

Surds such as $\sqrt{75}$ and $\sqrt[5]{x^8}$, whose coefficients are unity, are sometimes called *entire surds.*

Exercise 81.

Express as surds of the same lowest order

1. $\sqrt[5]{a^3}$, $\sqrt{a}$. 2. $\sqrt[4]{8}$, $\sqrt{3}$, $\sqrt[8]{6}$. 3. $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[6]{13}$.

Express as an entire surd

4. $11\sqrt{2}$. 5. $14\sqrt{5}$. 6. $6\sqrt[3]{4}$. 7. $5\sqrt[3]{6}$.

8. $\frac{4}{11}\sqrt{\frac{77}{8}}$. 9. $\frac{3ab}{2c}\sqrt{\frac{20c^2}{9a^2b}}$. 10. $\frac{2a}{3x}\sqrt[3]{\frac{27x^4}{a^2}}$.

Find the value of

11. $2\sqrt{15} \times 3\sqrt{5}$. 13. $\sqrt{12} \times \sqrt{27} \times \sqrt{75}$.

12. $8\sqrt{12} \times 3\sqrt{24}$. 14. $\sqrt[3]{16} \times \sqrt[3]{6} \times \sqrt[3]{9}$.

15. $\sqrt[3]{12} \times \sqrt[3]{75} \times \sqrt[3]{30}$.

16. $\sqrt[4]{6} \times \sqrt[4]{12} \times \sqrt[4]{18}$.

17. $\sqrt[3]{x+2} \times \sqrt[3]{x-2}$.

18. $\sqrt[3]{168} \times \sqrt[3]{147}$.

19. $5\sqrt[3]{128} \times 2\sqrt[3]{432}$.

20. $\sqrt{10} \times \sqrt[3]{200}$.

21. $\sqrt[3]{4} \times \sqrt{8}$.

22. $(\sqrt{6}-\sqrt{3})(\sqrt{6}+\sqrt{3})$.

23. $(2\sqrt{5}+3\sqrt{3})(3\sqrt{5}-4\sqrt{3})$.

24. $(\sqrt{2}+\sqrt{3}+\sqrt{6})(2\sqrt{2}+3\sqrt{3}+\sqrt{6})$.

25. $(\sqrt{2}+\sqrt{3}+\sqrt{5})^2$.

26. $(1+\sqrt[3]{3}+\sqrt[3]{5})(1+\sqrt[3]{3}-\sqrt[3]{5})$.

27. Which is the greater $\sqrt{3}$ or $\sqrt[3]{5\frac{1}{5}}$?

28. Arrange in order of magnitude $\sqrt{50}$, $\sqrt[3]{344}$, $\sqrt[4]{2402}$.

Simplify

29. $\sqrt[6]{49}$.

30. $\sqrt[4]{a^2b^2}$.

31. $\sqrt[6]{a^2b^2}$.

32. $\sqrt[8]{a^4b^4}$.

33. $\sqrt[8]{16a^4b^4}$.

34. $\sqrt[6]{27x^3y^3}$.

35. $\sqrt[6]{\dfrac{64x^2}{25y^2}}$.

36. $\sqrt[4]{\dfrac{25a^2}{(a+2)^2}}$.

252. When two binomials involving quadratic surds differ only in the quality of a surd term, they are said to be **Conjugate.**

Thus $3+\sqrt{2}$ and $3-\sqrt{2}$ are conjugate surds, so also are $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$, or $\sqrt{a}+\sqrt{b}$ and $-\sqrt{a}+\sqrt{b}$.

The product of two conjugate surds is rational.

Thus $(3+\sqrt{2})(3-\sqrt{2}) = 3^2-2 = 7$.

Also $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$.

253. The **quotient** of one surd by another is obtained in the simplest form by first multiplying both dividend and divisor by such a factor as will render the divisor rational. This process is called *rationalizing the divisor*, or *rationalizing the denominator*.

The cases that most frequently occur are the three following:

I. When the divisor is a monomial surd, as $\sqrt[n]{y^m}$, multiply both dividend and divisor by $\sqrt[n]{y^{n-m}}$.

Thus $$\frac{2}{3\sqrt{5}}=\frac{2\times\sqrt{5}}{3\sqrt{5}\times\sqrt{5}}=\frac{2}{15}\sqrt{5},$$

and $$\frac{b}{c\sqrt[5]{a^2}}=\frac{b\times\sqrt[5]{a^3}}{c\sqrt[5]{a^2}\times\sqrt[5]{a^3}}=\frac{b}{ca}\sqrt[5]{a^3}.$$

II. When the divisor is a binomial quadratic surd, multiply both dividend and divisor by the conjugate surd.

Thus $$\frac{5+\sqrt{7}}{3-\sqrt{7}}=\frac{(5+\sqrt{7})(3+\sqrt{7})}{(3-\sqrt{7})(3+\sqrt{7})}=\frac{22+8\sqrt{7}}{9-7}$$
$$=11+4\sqrt{7},$$

and $$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}$$
$$=\frac{a+2\sqrt{ab}+b}{a-b}.$$

III. When the divisor is of the form $(\sqrt{a}+\sqrt{b})+\sqrt{c}$, first multiply by its conjugate surd $(\sqrt{a}+\sqrt{b})-\sqrt{c}$.

The divisor thus becomes $(\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2$ or $(a+b-c)+2\sqrt{ab}$; then multiply by the conjugate surd $(a+b-c)-2\sqrt{ab}$. The divisor thus becomes the rational expression $(a+b-c)^2-(2\sqrt{ab})^2$.

Thus
$$\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}=\frac{\sqrt{2}(\sqrt{2}+\sqrt{3}+\sqrt{5})}{[(\sqrt{2}+\sqrt{3})-\sqrt{5}][(\sqrt{2}+\sqrt{3})+\sqrt{5}]}$$
$$=\frac{2+\sqrt{6}+\sqrt{10}}{(\sqrt{2}+\sqrt{3})^2-5}$$
$$=\frac{(2+\sqrt{6}+\sqrt{10})\times\sqrt{6}}{2\sqrt{6}\times\sqrt{6}}$$
$$=\frac{2\sqrt{6}+6+2\sqrt{15}}{12}=\tfrac{1}{6}(\sqrt{6}+3+\sqrt{15}).$$

Example. Given $\sqrt{5}=2.23607$, find the value of $3\div\sqrt{5}$.

$$\frac{3}{\sqrt{5}}=\frac{3\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\tfrac{3}{5}\sqrt{5}=\tfrac{3}{5}(2.23607)=1.34164.$$

Exercise 82.

Given $\sqrt{2}=1.41421$, $\sqrt{3}=1.73205$, $\sqrt{5}=2.23607$, $\sqrt{6}=2.44949$; find the value of

1. $14\div\sqrt{2}$.
2. $25\div\sqrt{5}$.
3. $48\div\sqrt{6}$.
4. $\sqrt{2}\div\sqrt{3}$.
5. $144\div\sqrt{6}$.
6. $4\div\sqrt{243}$.

Rationalize the divisor of

7. $\dfrac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$.
8. $\dfrac{15+14\sqrt{3}}{15-2\sqrt{3}}$.
9. $\dfrac{\sqrt{5}+3\sqrt{3}}{2\sqrt{5}-\sqrt{3}}$.
10. $\dfrac{\sqrt{6}-3\sqrt{12}}{2\sqrt{6}+\sqrt{12}}$.
11. $\dfrac{2\sqrt{3}+3\sqrt{2}}{5+2\sqrt{6}}$.
12. $\dfrac{\sqrt{9+x^2}-3}{\sqrt{9+x^2}+3}$.
13. $\dfrac{y^2}{x+\sqrt{x^2-y^2}}$.
14. $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$.

15. $\dfrac{3}{2+\sqrt{3}+\sqrt{5}}$.

16. $\dfrac{1+\sqrt{3}+\sqrt{6}}{1+\sqrt{2}-\sqrt{3}}$.

17. $\dfrac{\sqrt{3}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

18. $\dfrac{3}{\sqrt{2}-\sqrt{6}-\sqrt{7}}$.

19. $\dfrac{\sqrt{x-2}+\sqrt{x}}{\sqrt{x-2}-\sqrt{x}}$.

20. $\dfrac{a+\sqrt{a^2+3}}{a-\sqrt{a^2+3}}$.

21. $\dfrac{\sqrt{a-b}-\sqrt{a+b}}{\sqrt{a-b}+\sqrt{a+b}}$.

254. A power or root of a monomial surd is readily found by the theory of exponents.

EXAMPLE 1. Find the square of $3\sqrt[3]{ax}$.

$$(3\sqrt[3]{ax})^2 = 3^2(ax)^{\frac{2}{3}} = 9\sqrt[3]{a^2x^2}.$$

EXAMPLE 2. Find the cube of $2\sqrt{x}$.

$$(2\sqrt{x})^3 = 2^3x^{\frac{3}{2}} = 8xx^{\frac{1}{2}} = 8x\sqrt{x}.$$

EXAMPLE 3. Find the square root of $9a\sqrt{x^5y^5}$.

$$(9a\sqrt{x^5y^5})^{\frac{1}{2}} = (9ax^{\frac{5}{2}}y^{\frac{5}{2}})^{\frac{1}{2}} = 9^{\frac{1}{2}}a^{\frac{1}{2}}x^{\frac{5}{4}}y^{\frac{5}{4}}$$
$$= 3xya^{\frac{2}{4}}x^{\frac{1}{4}}y^{\frac{1}{4}} = 3xy\sqrt[4]{a^2xy}.$$

EXAMPLE 4. Find the cube root of $2a\sqrt{xy^3}$.

$$(2a\sqrt{xy^3})^{\frac{1}{3}} = (2ax^{\frac{1}{2}}y^{\frac{3}{2}})^{\frac{1}{3}} = 2^{\frac{1}{3}}a^{\frac{1}{3}}x^{\frac{1}{6}}y^{\frac{3}{6}}$$
$$= 2^{\frac{2}{6}}a^{\frac{2}{6}}x^{\frac{1}{6}}y^{\frac{3}{6}} = \sqrt[6]{4a^2xy^3}.$$

Exercise 83.

Perform as far as possible the operations indicated in

1. $(\sqrt[3]{a^2})^5$.

2. $(\sqrt[4]{a^3})^6$.

3. $(\sqrt[5]{b^3})^{11}$.

4. $(\sqrt[3]{a^2x^5})^8$.

5. $(2\sqrt{a^3bx})^4$.

6. $(3\sqrt[3]{a^2b^4x})^5$.

7. $(2\sqrt[3]{x^2-y^2})^6$. 10. $\sqrt[5]{2a\sqrt{xy}}$. 13. $\sqrt[3]{\sqrt[4]{81a^{24}}}$.

8. $\sqrt[3]{\sqrt{(x-y)^7}}$. 11. $\sqrt[3]{\sqrt{64}}$. 14. $\sqrt[3]{16\sqrt{2a}}$.

9. $\sqrt[4]{\sqrt[3]{(c-d)^{15}}}$. 12. $\sqrt[4]{\sqrt[3]{625}}$. 15. $\sqrt[4]{b^2cd^3\sqrt[3]{a^2cd}}$.

PROPERTIES OF QUADRATIC SURDS.

255. *A quadratic surd cannot equal the sum of a rational number and a surd.*

For if possible let $\sqrt{a}=n+\sqrt{m};$ (1)

then $a=n^2+m+2n\sqrt{m},$

or $2n\sqrt{m}=a-n^2-m,$

which is impossible, since a surd cannot equal a rational expression; hence (1) is impossible.

256. *If* $a+\sqrt{b}=x+\sqrt{y}$, *then* a *will equal* x, *and* b *will equal* y.

For if possible let $a=x+m;$

then $x+m+\sqrt{b}=x+\sqrt{y},$

or $\sqrt{y}=\sqrt{b}+m,$

which is impossible by § 255.

Hence $a=x,$

and therefore $\sqrt{b}=\sqrt{y}$, or $b=y$.

257. *To extract the square root of a binomial surd.*

Suppose $\sqrt{a\pm\sqrt{b}}=\sqrt{x}\pm\sqrt{y}.$ (1)

Square, $a\pm\sqrt{b}=x+y\pm2\sqrt{xy}.$

Hence by § 256 we have

$$x+y=a, \text{ and } 2\sqrt{xy}=\sqrt{b},$$

or $x+y=a$, and $4xy=b$. (2)

Solving equations (2) for x and y, and substituting their values in (1), we obtain the value of $\sqrt{a \pm \sqrt{b}}$.

We shall here consider only those cases in which equations (2) may be solved by inspection.

EXAMPLE 1. Find the square root of $18 + 8\sqrt{5}$.

Assume $\sqrt{18 + 8\sqrt{5}} = \sqrt{x} + \sqrt{y}$. (1)

Square $18 + 8\sqrt{5} = x + y + 2\sqrt{xy}$.

$\therefore\ x + y = 18$, and $2\sqrt{xy} = 8\sqrt{5}$; § 256

or $x + y = 18$, and $xy = 80$.

$\therefore\ x = 8$, and $y = 10$.

$$\therefore\ \sqrt{18 + 8\sqrt{5}} = \sqrt{8} + \sqrt{10} = 2\sqrt{2} + \sqrt{10}.$$

EXAMPLE 2. Extract the square root of $83 - 12\sqrt{35}$.

Assume $\sqrt{83 - 12\sqrt{35}} = \sqrt{x} - \sqrt{y}$.

Square $83 - 12\sqrt{35} = x + y - 2\sqrt{xy}$.

$\therefore\ x + y = 83$, and $2\sqrt{xy} = 12\sqrt{35}$,

or $x + y = 83$, and $xy = 1260$.

$\therefore\ x = 63$, and $y = 20$.

$$\therefore\ \sqrt{83 - 12\sqrt{35}} = \sqrt{63} - \sqrt{20} = 3\sqrt{7} - 2\sqrt{5}.$$

Exercise 84.

Find the square root of

1. $6 + \sqrt{20}$.
2. $12 - 6\sqrt{3}$.
3. $16 + 6\sqrt{7}$.
4. $4\frac{1}{2} + 2\sqrt{2}$.
5. $28 - 5\sqrt{12}$.
6. $117 + 36\sqrt{10}$.
7. $7 - 2\sqrt{10}$.
8. $75 + 12\sqrt{21}$.
9. $47 - 4\sqrt{33}$.
10. $2\frac{1}{4} + \sqrt{5}$.
11. $4\frac{1}{8} - \frac{4}{8}\sqrt{3}$.
12. $16 + 5\sqrt{7}$.
13. $19 + 8\sqrt{3}$.
14. $11 + 4\sqrt{6}$.
15. $15 - 4\sqrt{14}$.

IMAGINARY EXPRESSIONS.

258. An **Imaginary Expression** is an expression that involves an imaginary number.

In this chapter we shall consider only *quadratic* imaginary expressions; that is, such as involve only the *square root* of a negative number.

NOTE. By the methods of Higher Algebra or Trigonometry any imaginary number may be expressed in terms of a quadratic imaginary number. An imaginary expression is sometimes called a *Complex Expression*.

259. To *add* or *subtract* quadratic imaginary expressions, reduce each imaginary term to the form $c\sqrt{-1}$ (§ 139), and proceed as in the case of surds.

260. *To find the successive powers of* $\sqrt{-1}$.

By § 120, $(\sqrt{-1})^2 = -1 \qquad = -1.$

$$\therefore\ (\sqrt{-1})^3 = (-1)(\sqrt{-1}) = -\sqrt{-1}.$$

$$\therefore\ (\sqrt{-1})^4 = (-1)(-1) = +1.$$

Hence if n denote any positive integer, we have

$$(\sqrt{-1})^{4n} = \{(\sqrt{-1})^4\}^n = (+1)^n = +1;$$

$$\therefore\ (\sqrt{-1})^{4n+1} = \sqrt{-1};$$

$$\therefore\ (\sqrt{-1})^{4n+2} = -1;$$

$$\therefore\ (\sqrt{-1})^{4n+3} = -\sqrt{-1}.$$

Thus $\qquad (\sqrt{-1})^7 = (\sqrt{-1})^4(\sqrt{-1})^3 = -\sqrt{-1},$

and $\qquad (\sqrt{-1})^{10} = (\sqrt{-1})^8(\sqrt{-1})^2 = -1.$

261. The **Typical Form** of quadratic imaginary expressions is $a + b\sqrt{-1}$, in which a and b are real numbers.

If $a = 0$, the expression contains no real term and is called a **Pure Imaginary.**

Any imaginary expression may be made to assume the typical form.

Thus $7 + \sqrt{-12} = 7 + 2\sqrt{3}\sqrt{-1}$.

Here $a = 7$, and $b = 2\sqrt{3}$.

262. Two typical imaginary expressions are said to be **Conjugate,** when they differ only in the quality of the imaginary term.

Thus $a - b\sqrt{-1}$ and $a + b\sqrt{-1}$ are conjugate imaginary expressions; so, also, are $4 + 3\sqrt{-1}$ and $4 - 3\sqrt{-1}$.

263. *The sum and product of two conjugate imaginary expressions are both real.*

For $a + b\sqrt{-1} + (a - b\sqrt{-1}) = 2a$;

and
$$(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 - (-b^2)$$
$$= a^2 + b^2.$$

264. *If two imaginary expressions are equal, the real parts are equal and the imaginary parts are equal.*

For let $a + b\sqrt{-1} = x + y\sqrt{-1}$;

then $a - x = (y - b)\sqrt{-1}$.

which is impossible unless $a = x$ and $y = b$.

265. If $a + b\sqrt{-1} = 0$; then, by § 264, $a = 0$ and $b = 0$.

266. To multiply or divide one imaginary expression by another, reduce them each to the typical form, then proceed as in the multiplication or division of surds, obtaining the product of the imaginary factors by § 260.

Example 1. Multiply $3+\sqrt{-5}$ by $4-\sqrt{-3}$.

$$\begin{array}{l}
3+\ \ \sqrt{5}\sqrt{-1} \\
\underline{4-\ \ \sqrt{3}\sqrt{-1}} \\
12+\ 4\sqrt{5}\sqrt{-1} \\
\underline{\ -\ 3\sqrt{3}\sqrt{-1}-\sqrt{15}\,(-1)} \\
12+(4\sqrt{5}-3\sqrt{3})\sqrt{-1}+\sqrt{15}
\end{array}$$

Example 2. Divide $\sqrt{-7}$ by $\sqrt{-5}$.

$$\frac{\sqrt{-7}}{\sqrt{-5}}=\frac{\sqrt{7}\times\sqrt{-1}}{\sqrt{5}\times\sqrt{-1}}=\frac{\sqrt{7}}{\sqrt{5}}=\tfrac{1}{5}\sqrt{35}.$$

Example 3. Divide $4+3\sqrt{-1}$ by $3-2\sqrt{-1}$.

$$\begin{aligned}
\frac{4+3\sqrt{-1}}{3-2\sqrt{-1}}&=\frac{(4+3\sqrt{-1})(3+2\sqrt{-1})}{(3-2\sqrt{-1})(3+2\sqrt{-1})}\\
&=\frac{6+17\sqrt{-1}}{9+4}\\
&=\tfrac{6}{13}+\tfrac{17}{13}\sqrt{-1}.
\end{aligned}$$

Exercise 85.

Simplify

1. $\sqrt{-9}+\sqrt{-4}-\sqrt{-16}+\sqrt{-25}$.
2. $\sqrt{-36}-\sqrt{-49}-\sqrt{-1}+\sqrt{-64}$.
3. $\sqrt{-81}+\sqrt{-100}-\sqrt{-121}-\sqrt{-49}$.
4. $\sqrt{-4a^2}+\sqrt{-36a^2}-\sqrt{-a^2}+\sqrt{-9a^2}$.
5. $a+\sqrt{-b}+4a-\sqrt{-b}-2a$.
6. $\sqrt{-3}+\sqrt{-12}+\sqrt{-27}-\sqrt{-48}$.
7. $\sqrt{-3a}+\sqrt{-27a}+\sqrt{-75a}-\sqrt{-108a}$.

Multiply

8. $\sqrt{-4}$ by $\sqrt{-16}$.
9. $\sqrt{-5}$ by $\sqrt{-7}$.

10. $-\sqrt{-7}$ by $\sqrt{-11}$. **11.** $-\sqrt{-16}$ by $-\sqrt{-36}$.

12. $\sqrt{-a^2}$ by $\sqrt{-a^3}$.

13. $-\sqrt{-a^2b^3}$ by $\sqrt{-a^4b^3}$.

14. $\sqrt{-4}\times\sqrt{-9}\times\sqrt{-16}\times\sqrt{-25}$ by $\sqrt{-36}$.

15. $\sqrt{-a}\times\sqrt{-a^3}\times\sqrt{-a^2}\times\sqrt{-a^3}\times\sqrt{-a^5}$ by $\sqrt{-a^4}$.

16. $\sqrt{-3}+\sqrt{-5}$ by $\sqrt{-2}-\sqrt{-4}$.

17. $a+\sqrt{-3}$ by $-b-\sqrt{-7}$.

18. $2\sqrt{-3}+5\sqrt{-7}$ by $3\sqrt{-5}-4\sqrt{-9}$.

19. $x\sqrt{-x}+y\sqrt{-y}$ by $x\sqrt{-x}-y\sqrt{-y}$.

20. $\sqrt{-7}+5\sqrt{-3}$ by $\sqrt{-7}-5\sqrt{-3}$.

21. $\sqrt{-a}+b\sqrt{-c}$ by $\sqrt{-a}-b\sqrt{-c}$.

Find the quotient of

22. $\dfrac{2}{\sqrt{-1}}$.

23. $\dfrac{x}{\sqrt{-4}}$.

24. $\dfrac{y}{\sqrt{-x^2}}$.

25. $\dfrac{\sqrt{-16}}{\sqrt{-25}}$.

26. $\dfrac{\sqrt{-ay^2}}{\sqrt{-y^2}}$.

27. $\dfrac{-a}{\sqrt{-4a^2}}$.

28. $\dfrac{\sqrt{-a^2}}{-a\sqrt{-a}}$.

29. $\dfrac{\sqrt{-81}}{-\sqrt{-64}}$.

30. $\dfrac{\sqrt{-8x^3}}{-\sqrt{-27x^2}}$.

31. $\dfrac{7\sqrt{-a^2}}{-2\sqrt{-a}}$.

32. $\dfrac{1}{2-\sqrt{-3}}$.

33. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$.

34. $\dfrac{4+\sqrt{-2}}{2-\sqrt{-2}}$.

35. $\dfrac{1}{3-2\sqrt{-3}}$.

36. $\dfrac{a+\sqrt{-x}}{a-\sqrt{-x}}$.

CHAPTER XXI.

EQUATIONS INVOLVING SURDS, AND HIGHER EQUATIONS.

267. *If both members of an equation be raised to the same power, in general new solutions will be introduced.*

Let $$A = B \tag{1}$$

be any equation; then the equation

$$A^2 = B^2, \text{ or } A^2 - B^2 = 0,$$

is equivalent to the two equations

$$A - B = 0 \text{ and } A + B = 0.$$

Hence the solutions of $A + B = 0$ would be introduced by squaring both members of (1).

Therefore, if in the process of solving an equation both its members were squared, those solutions of the resulting equation that do not satisfy the given equation were introduced by squaring, and should be rejected.

268. In some equations the unknown letter appears under the radical sign. The simpler cases of such equations can generally be solved as follows: Bring the more complex surd term to one member by itself, and then square both members. Repeat this process until the resulting equation is rational.

Example 1. Solve $$2\sqrt{x}-\sqrt{4x-11}=1. \quad (1)$$

Transpose, $$2\sqrt{x}-1=\sqrt{4x-11}.$$

Square, $$4x-4\sqrt{x}+1=4x-11.$$

$$\therefore\ \sqrt{x}=3.$$

$$\therefore\ x=9.$$

Since $x=9$ satisfies (1), 9 is the solution of (1), and no solution was introduced by squaring (1).

Example 2. Solve $$\sqrt{4-x}=x-4. \quad (1)$$

Square, $$4-x=x^2-8x+16.$$

Transpose, $$x^2-7x+12=0.$$

Factor, $$(x-3)(x-4)=0. \quad (2)$$

The solutions of (2) are evidently 3 and 4. Now $x=4$ satisfies (1), but $x=3$ does not. Hence the solution 3 was introduced by squaring (1), and 4 is the only solution of (1).

Example 3. Solve $$2-\sqrt{2x+8}+2\sqrt{x+5}=0. \quad (1)$$

Transpose, $$2+2\sqrt{x+5}=\sqrt{2x+8}.$$

Square, $$4+8\sqrt{x+5}+4x+20=2x+8.$$

Transpose, $$x+8=-4\sqrt{x+5}.$$

Square, $$x^2+16x+64=16x+80.$$

Transpose, $$x^2-16=0. \quad (2)$$

The solutions of (2) are 4 and -4. Hence, if (1) has any solution, it must be 4 or -4. But neither of these solutions satisfy (1); hence (1) has no solution, that is, it is impossible.

Both solutions of (2) were introduced by squaring.

It should be noted, however, that if we use both the positive and the negative values of $\sqrt{2x+8}$ and $\sqrt{x+5}$, we obtain in addition to (1) the three equations,

$$2-\sqrt{2x+8}-2\sqrt{x+5}=0, \quad (3)$$

$$2+\sqrt{2x+8}-2\sqrt{x+5}=0, \quad (4)$$

$$2+\sqrt{2x+8}+2\sqrt{x+5}=0. \quad (5)$$

Multiplying together the first members of (1), (3), (4), and (5), we obtain the first member of (2); hence equation (2) is equivalent to the four equations (1), (3), (4), and (5). Now by trial we find that -4 satisfies (3), and 4 satisfies (4), but neither 4 nor -4 will satisfy (5); hence (5) is impossible as well as (1).

Equation (2) could be obtained from (3), (4), or (5) in the same way that it was from (1).

269. Some equations involving surds may be solved as below.

EXAMPLE. Solve $2x^2 + 3x - 5\sqrt{2x^2 + 3x + 9} = -3.$ (1)

Add 9, $(2x^2 + 3x + 9) - 5\sqrt{2x^2 + 3x + 9} = 6.$ (2)

Since $2x^2 + 3x + 9$ is the square of $\sqrt{2x^2 + 3x + 9}$, the value of $\sqrt{2x^2 + 3x + 9}$ may be written out by § 185.

Thus $\sqrt{2x^2 + 3x + 9} = \frac{5}{2} \pm \sqrt{6 + \frac{25}{4}} = 6$, or -1;

that is, $\sqrt{2x^2 + 3x + 9} = 6,$ (3)

and $\sqrt{2x^2 + 3x + 9} = -1,$ (4)

are together equivalent to equation (1).

The solutions of (3) are 3 and $-4\frac{1}{2}$; while (4) is evidently an impossible equation, since no positive number can equal -1. Hence the solutions of (1) are 3 and $-4\frac{1}{2}$.

Exercise 86.

Solve

1. $\sqrt{x - 5} = 3.$
2. $7 - \sqrt{x - 4} = 3.$
3. $\sqrt{5x - 1} = 2\sqrt{x + 3}.$
4. $2\sqrt{3 - 7x} = 3\sqrt{8x - 12}.$
5. $\sqrt{9x^2 - 11x - 5} = 3x - 2.$
6. $\sqrt{4x^2 - 7x + 1} = 2x - 1\frac{4}{5}.$
7. $\sqrt{x + 25} = 1 + \sqrt{x}.$
8. $\sqrt{x + 3} + \sqrt{x} = 5.$
9. $\sqrt{8x + 33} - 3 = 2\sqrt{2x}.$
10. $10 - \sqrt{25 + 9x} = 3\sqrt{x}.$
11. $\sqrt{9x - 8} = 3\sqrt{x + 4} - 2.$
12. $\sqrt{x - 4} + 3 = \sqrt{x + 11}.$

13. $\sqrt{4x+5}-\sqrt{x}=\sqrt{x+3}.$

14. $\sqrt{8x+17}-\sqrt{2x}=\sqrt{2x+9}.$

15. $\sqrt{3x-11}+\sqrt{3x}=\sqrt{12x-23}.$

16. $\sqrt{12x-5}+\sqrt{3x-1}=\sqrt{27x-2}.$

17. $\sqrt{x+3}+\sqrt{x+8}=\sqrt{4x+21}.$

18. $\sqrt{x+2}+\sqrt{4x+1}=\sqrt{9x+7}.$

19. $\sqrt{x+4ab}=2a+\sqrt{x}.$

20. $\sqrt{x}+\sqrt{4a+x}=2\sqrt{b+x}.$

21. $\sqrt{x-1}+\sqrt{x}=2\div\sqrt{x}.$

22. $\sqrt{x+5}+\sqrt{x}=10\div\sqrt{x}.$

23. $\sqrt{x}-\sqrt{x-8}=2\div\sqrt{x-8}.$

24. $\sqrt{1+x}+\sqrt{x}=2\div\sqrt{1+x}.$

25. $2\sqrt{x}-\sqrt{4x-3}=1\div\sqrt{4x-3}.$

26. $\sqrt{x}-7=1\div(\sqrt{x}+7).$

27. $\dfrac{x-1}{\sqrt{x}-1}=3+\dfrac{\sqrt{x}+1}{2}.$

28. $\dfrac{2x-3}{\sqrt{x-2}+1}=2\sqrt{x-2}-1.$

29. $2=\dfrac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}}.$

30. $\dfrac{1}{1-x}+\dfrac{1}{\sqrt{x}+1}+\dfrac{1}{\sqrt{x}-1}=0.$

31. $3x^2-4x+\sqrt{3x^2-4x-6}=18.$

32. $x^2 - x + 4 + \sqrt{x^2 - x + 4} = 2.$

33. $x^2 + 2x - \sqrt{x^2 + 2x - 6} = 12.$

34. $1 + \frac{1}{5}\sqrt{x^2 + x + 5} = \dfrac{10}{\sqrt{x^2 + x + 5}}.$

35. $x^2 + \sqrt{4x^2 + 24x} = 24 - 6x.$

36. $2x^2 + 6x = 1 - \sqrt{x^2 + 3x + 1}.$

37. $2(2x - 3)(x - 4) - \sqrt{2x^2 - 11x + 15} = 60.$

38. $\dfrac{\sqrt{x} + 3}{\sqrt{x} - 2} = \dfrac{3\sqrt{x} - 5}{3\sqrt{x} - 13}.$

39. $\dfrac{9\sqrt{x} - 23}{3\sqrt{x} - 8} = \dfrac{6\sqrt{x} - 17}{2\sqrt{x} - 6}.$

40. $2 - \dfrac{\sqrt{x} + 3}{\sqrt{x} + 2} = \dfrac{\sqrt{x} + 9}{\sqrt{x} + 7}.$

41. $\dfrac{6\sqrt{x} - 7}{\sqrt{x} - 1} - 5 = \dfrac{7\sqrt{x} - 26}{7\sqrt{x} - 21}.$

42. $\dfrac{2\sqrt{x} - 1}{2\sqrt{x} + \frac{4}{3}} = \dfrac{\sqrt{x} - 2}{\sqrt{x} - \frac{4}{3}}.$

270. Sometimes equations of a higher degree than the second can be solved by factoring or by the methods of solving quadratics.

EXAMPLE 1. Solve $x^3 = 1$, or $x^3 - 1 = 0$. (1)

Factor, $(x - 1)(x^2 + x + 1) = 0$. (2)

Equation (2) is equivalent to the two equations

$x - 1 = 0$, and $x^2 + x + 1 = 0$. (3)

The solutions of equations (3) are 1 and $-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$. Hence equation (1) has three solutions, one being real and the other two imaginary. Therefore there are three different numbers the cubes of which each equal 1; that is, 1 has three cube roots. In like manner it can be shown that any number has three cube roots.

Example 2. Solve $x^4 = 1$, or $x^4 - 1 = 0$. (1)

Factor, $(x^2 - 1)(x^2 + 1) = 0$. (2)

Equation (2) is equivalent to the two equations

$$x^2 - 1 = 0, \text{ and } x^2 + 1 = 0. \qquad (3)$$

The solutions of (3) are ± 1 and $\pm\sqrt{-1}$.

Hence 1 has four fourth roots, two real and two imaginary. So also has any other *positive* number.

Example 3. Solve $x^5 = 1$, or $x^5 - 1 = 0$. (1)

Factor, $(x - 1)(x^4 + x^3 + x^2 + x + 1) = 0$. (2)

One solution of (2) is 1, and the other solutions are those of the equation

$$x^4 + x^3 + x^2 + x + 1 = 0. \qquad (3)$$

Divide by x^2, $x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0$.

Add 1, $x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} = 1$,

or $\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 1$.

$$\therefore\ x + \frac{1}{x} = -\tfrac{1}{2} \pm \sqrt{1 + \tfrac{1}{4}} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}. \qquad \S\ 185$$

$$\therefore\ x^2 + 1 = \tfrac{1}{2}(-1 \pm \sqrt{5})\,x. \qquad (4)$$

Solving the two quadratics in (4), we obtain four solutions, all of which are imaginary. Hence 1, or any other real number, has five fifth roots, one of which is real, and the other four imaginary.

Example 4. Solve $x^6 = 1$, or $x^6 - 1 = 0$. (1)

Factor, $(x^3 - 1)(x^3 + 1) = 0$,

or $(x-1)(x^2+x+1)(x+1)(x^2-x+1) = 0$. (2)

Equation (2) is equivalent to the four equations

$$x - 1 = 0,\ x^2 + x + 1 = 0,\ x + 1 = 0,\ x^2 - x + 1 = 0. \qquad (3)$$

Solving equations (3), we obtain six solutions, two real and four imaginary. Hence any *positive* number has six sixth roots, two real and four imaginary.

Example 5. Solve $x^8 = 1$, or $x^8 - 1 = 0$. (1)

Factor, $(x^4 + 1)(x^2 + 1)(x^2 - 1) = 0$. (2)

Equation (2) is equivalent to the three equations

$$x^2 + 1 = 0,\ x^2 - 1 = 0,\ x^4 + 1 = 0. \quad (3)$$

To solve $x^4 + 1 = 0$, add and subtract $2x^2$,

$$x^4 + 2x^2 + 1 - 2x^2 = 0.$$

Factor, $(x^2 + 1 + x\sqrt{2})(x^2 + 1 - x\sqrt{2}) = 0$. (4)

Equation (4) is equivalent to the two equations

$$x^2 + 1 + x\sqrt{2} = 0,$$

and $x^2 + 1 - x\sqrt{2} = 0$, (5)

each of which has two solutions.

Hence any *positive* number has eight eighth roots, two real and six imaginary.

Example 6. Solve $(x^2 + 2x)^2 - 5(x^2 + 2x) - 14 = 0$. (1)

Factor, $(x^2 + 2x - 7)(x^2 + 2x + 2) = 0$.

Example 7. Solve $\dfrac{x^2}{x-1} + \dfrac{x-1}{x^2} = \dfrac{17}{4}$. (1)

Here the second term is the reciprocal of the first. Putting y for the first term, and the reciprocal of y for the second, (1) becomes

$$y + \frac{1}{y} = \frac{17}{4}.$$

Multiply by $4y$, $4y^2 - 17y + 4 = 0$.

Factor, $(y - 4)(4y - 1) = 0$.

$$\therefore\ y = 4, \text{ or } \tfrac{1}{4}.$$

Hence (1) is equivalent to the two equations

$$\frac{x^2}{x-1} = 4 \text{ and } \frac{x^2}{x-1} = \frac{1}{4}. \quad (2)$$

The solutions of equations (2) are 2, 2, $\frac{1}{8}(1 \pm \sqrt{-15})$.

Exercise 87.

Solve

1. $x^4 - 5x^2 + 4 = 0.$

2. $x^4 - 10x^2 + 9 = 0.$

3. $x^4 - 7x^2 - 18 = 0.$

4. $\frac{x^2 - 1}{9} + \frac{1}{x^2} = 1.$

5. $x^2 + \frac{100}{x^2} = 29.$

6. $(x^2 + x)^2 - 22(x^2 + x) = -40.$

7. $(x^2 - x)^2 - 8(x^2 - x) = -12.$

8. $\left(x + \frac{1}{x}\right)^2 + 4\left(x + \frac{1}{x}\right) = 12.$

9. $2x^2 - 4x - \sqrt{x^2 - 2x - 3} = 9.$

10. $\frac{x^2}{x + 1} + \frac{x + 1}{x^2} = 2.$

11. $\frac{x}{x^2 + 1} + \frac{x^2 + 1}{x} = \frac{5}{2}.$

12. $\frac{x^2 + 2}{x^2 + 4x + 1} + \frac{x^2 + 4x + 1}{x^2 + 2} = \frac{5}{2}.$

13. $x^2 + x + 1 = \frac{42}{x^2 + x}.$

14. $x^3 + 1 = 0.$

15. $x^3 - 27 = 0.$

16. $x^5 + 1 = 0.$

17. $x^6 + 1 = 0.$

18. $x^3 + 8 = 0.$

19. $x^4 + 16 = 0.$

20. $x^6 - 64 = 0.$

21. $x^4 - 7x + 12 = 0.$

CHAPTER XXII.

SYSTEMS OF QUADRATIC AND HIGHER EQUATIONS.

271. A system consisting of one linear and one quadratic equation in x and y has in general two, and only two, solutions.

Let the system be,
$$\left.\begin{aligned} x + 2y &= 5, &(1)\\ x^2 + 2y^2 &= 9. &(2)\end{aligned}\right\}(a)$$

Solve (1) for x, From (2) and (3),
$$\left.\begin{aligned} x &= 5 - 2y. &(3)\\ (5-2y)^2 + 2y^2 &= 9. &\\ \therefore\ 6y^2 - 20y + 16 &= 0. &\\ \therefore\ 3y^2 - 10y + 8 &= 0. &\\ \therefore\ (3y-4)(y-2) &= 0. &(4)\end{aligned}\right\}(b)$$

System (b) is equivalent to the two systems
$$\left.\begin{aligned} x &= 5 - 2y,\\ 3y - 4 &= 0.\end{aligned}\right\}(c) \qquad \left.\begin{aligned} x &= 5 - 2y,\\ y - 2 &= 0.\end{aligned}\right\}(d)$$

The solution of (c) is $x = \frac{7}{3}$, $y = \frac{4}{3}$; and that of (d) is $x = 1$, $y = 2$. By § 205 systems (a) and (b) are equivalent; hence the solutions of (c) and (d) are the two solutions of (a).

The two solutions of system (a) may be written briefly 1, 2; and $\frac{7}{3}$, $\frac{4}{3}$; provided it be understood that in each solution the value of x is placed first.

Exercise 88.

Solve the system

1. $\left.\begin{aligned} x + y &= 15\\ xy &= 36\end{aligned}\right\}$

2. $\left.\begin{aligned} x + y &= 51\\ xy &= 518\end{aligned}\right\}$

3. $\left.\begin{aligned} 8x - 4y &= -12\\ 3x^2 + 2y^2 - y &= 48\end{aligned}\right\}$

4. $\left.\begin{aligned} x - y &= 10\\ x^2 + y^2 &= 58\end{aligned}\right\}$

5. $\left.\begin{array}{l} 3x+3y=10 \\ xy=1 \end{array}\right\}$

6. $\left.\begin{array}{l} 2x-5y=0 \\ x^2-3y^2=13 \end{array}\right\}$

7. $\left.\begin{array}{l} 2x+3y=0 \\ 4x^2+9xy+9y^2=72 \end{array}\right\}$

8. $\left.\begin{array}{l} x-y=3 \\ x^2+19+y^2=3xy \end{array}\right\}$

9. $\left.\begin{array}{l} 2x-y=5 \\ x+3y=2xy \end{array}\right\}$

10. $\left.\begin{array}{l} 3x+2y=5 \\ x^2-4xy+5y^2=2 \end{array}\right\}$

11. $\left.\begin{array}{l} 3x^2-2xy=15 \\ 2x+3y=12 \end{array}\right\}$

12. $\left.\begin{array}{l} x+y=15 \\ x^2+y^2=125 \end{array}\right\}$

13. $\left.\begin{array}{l} x^2+3xy-y^2=23 \\ x+2y=7 \end{array}\right\}$

14. $\left.\begin{array}{l} x^2+y^2=185 \\ x-y=3 \end{array}\right\}$

15. $\left.\begin{array}{l} 2x-7y=25 \\ 5x^2+4xy+3y^2=23 \end{array}\right\}$

16. $\left.\begin{array}{l} 3x-31=5y \\ x^2+5xy+25=y^2 \end{array}\right\}$

17. $\left.\begin{array}{l} x+y=2 \\ \frac{2}{x}+\frac{3}{y}=6 \end{array}\right\}$

18. $\left.\begin{array}{l} x+2y=7 \\ \frac{3}{x}+\frac{6}{y}=5 \end{array}\right\}$

19. $\left.\begin{array}{l} x-y=1 \\ \frac{x}{y}-\frac{y}{x}=\frac{5}{6} \end{array}\right\}$

20. $\left.\begin{array}{l} 2x-y=4 \\ \frac{3}{x}+\frac{4}{y}=1 \end{array}\right\}$

272. A system of two quadratic equations in x and y has, in general, four, and only four, solutions.

Let the system be

$$\left.\begin{array}{lr} x^2+y=a, & (1) \\ x+y^2=b. & (2) \end{array}\right\}(a)$$

From (1), $y=a-x^2$. (3)

From (2) and (3), $x+(a-x^2)^2=b$,

or $x^4-2ax^2+x+a^2-b=0$. (4)

$$\left.\begin{array}{lr} y=a-x^2. & (3) \\ x+(a-x^2)^2=b, & \\ x^4-2ax^2+x+a^2-b=0. & (4) \end{array}\right\}(b)$$

By § 205 system (b) is equivalent to (a).

In equation (4), x has four values; and in (3), y has one value for each value of x; hence system (b), and therefore system (a), has four solutions.

Here, as is usually the case, the elimination of y between the two quadratic equations leads to an equation of the fourth degree in x, which in general cannot be solved by any method within the range of this book. But many systems of quadratic and higher equations can be solved by the methods of solving quadratic equations. A few of the most frequently occurring cases will be considered below.

273. Systems in which all the unknown terms are of the second degree may be solved as below:

Example 1. Solve
$$\left.\begin{aligned} x^2+xy+2y^2&=44, &(1)\\ 2x^2-xy+y^2&=16. &(2)\end{aligned}\right\}(a)$$

Multiply (1) by 4, $\quad 4x^2+4xy+8y^2=176. \quad (3)$

Multiply (2) by 11, $\quad 22x^2-11xy+11y^2=176. \quad (4)$

Subtract (3) from (4), $\quad 18x^2-15xy+3y^2=0. \quad (5)$

Factor, $\quad (y-3x)(y-2x)=0. \quad (6)$

Equations (6) and (1) form a system that is equivalent to the two systems

$$\left.\begin{aligned} x^2+xy+2y^2&=44,\\ y-3x&=0.\end{aligned}\right\} \qquad \left.\begin{aligned} x^2+xy+2y^2&=44,\\ y-2x&=0.\end{aligned}\right\} \qquad (b)$$

Systems (b) are readily solved by the method of § 271.

The solutions of systems (b) are $\sqrt{2}$, $3\sqrt{2}$; $-\sqrt{2}$, $-3\sqrt{2}$; 2, 4; and -2, -4; which are therefore the four solutions of (a).

We multiplied (1) and (2) by the smallest numbers that would make the known terms in (3) and (4) equal.

Example 2. Solve
$$\left.\begin{aligned} y^2-2x^2&=4x, &(1)\\ 3y^2+xy-2x^2&=16x. &(2)\end{aligned}\right\}(a)$$

The method employed in Example 1 is applicable to this system also.

Multiply (1) by 4, $\quad 4y^2-8x^2=16x. \quad (3)$

Subtract (2) from (3), $\quad y^2-xy-6x^2=0.$

Factor, $\quad (y+2x)(y-3x)=0. \quad (4)$

Equations (4) and (1) form a system that is equivalent to system (a) and also to the two following systems (b) and (c).

$$\left.\begin{aligned} y^2-2x^2&=4x,\\ y+2x&=0,\end{aligned}\right\}(b) \qquad \left.\begin{aligned} y^2-2x^2&=4x,\\ y-3x&=0,\end{aligned}\right\}(c)$$

Systems (*b*) and (*c*) are readily solved.

The solutions of system (*b*) are 0, 0 and 2, -4; those of system (*c*) are 0, 0 and $\frac{4}{7}$, $\frac{12}{7}$. These four solutions are therefore the four solutions of system (*a*).

In each of the foregoing examples we combined the given equations in such a way as to obtain a homogeneous equation of the second degree.

EXAMPLE 2. Solve
$$\left.\begin{aligned} x^2-2xy&=0, &(1)\\ 4x^2+9y^2&=225. &(2)\end{aligned}\right\}(a)$$

Factor (1),
$$x(x-2y)=0. \qquad (3)$$

Hence system (*a*) is equivalent to the two systems

$$\left.\begin{aligned} 4x^2+9y^2&=225,\\ x&=0,\end{aligned}\right\} \qquad \left.\begin{aligned} 4x^2+9y^2&=225,\\ x-2y&=0,\end{aligned}\right\}$$

which are readily solved.

The four solutions are 0, 5; 0, -5; 6, 3; and -6, -3.

Exercise 89.

Solve

1. $\left.\begin{aligned} x^2-3xy&=0\\ 5x^2+3y^2&=48\end{aligned}\right\}$

2. $\left.\begin{aligned} 2x^2-3xy&=0\\ y^2+5xy&=34\end{aligned}\right\}$

3. $\left.\begin{aligned} x^2+xy&=12\\ xy-y^2&=2\end{aligned}\right\}$

4. $\left.\begin{aligned} x^2-3xy&=10\\ 4y^2-xy&=-1\end{aligned}\right\}$

5. $\left.\begin{aligned} x^2+xy&=24\\ 2y^2+3xy&=32\end{aligned}\right\}$

6. $\left.\begin{aligned} x^2+3xy&=7\\ y^2+xy&=6\end{aligned}\right\}$

7. $\left.\begin{aligned} 3xy+x^2&=10\\ 5xy-2x^2&=2\end{aligned}\right\}$

8. $\left.\begin{aligned} 3x^2-5y^2&=28\\ 3xy-4y^2&=8\end{aligned}\right\}$

9. $\left.\begin{aligned} x^2+3xy&=28\\ xy+4y^2&=8\end{aligned}\right\}$

10. $\left.\begin{aligned} x^2-3xy+2y^2&=3\\ 2x^2+y^2&=6\end{aligned}\right\}$

11. $$\left.\begin{aligned} x^2+5y^2&=84\\ 3x^2+17xy+84&=y^2 \end{aligned}\right\}$$

12. $$\left.\begin{aligned} x^2-7xy-9y^2&=9\\ x^2+5xy+11y^2&=5 \end{aligned}\right\}$$

13. $$\left.\begin{aligned} x^2-2xy+5&=0\\ (x-y)^2-4&=0 \end{aligned}\right\}$$

14. $$\left.\begin{aligned} x(x+y)&=40\\ y(x-y)&=\ \ 6 \end{aligned}\right\}$$

15. $$\left.\begin{aligned} x^2+3xy&=\ \ 54\\ xy+4y^2&=115 \end{aligned}\right\}$$

16. $$\left.\begin{aligned} x^2+xy+44&=2y^2\\ xy+3y^2&=80 \end{aligned}\right\}$$

17. $$\left.\begin{aligned} x^2+xy+y^2&=7\\ 6x^2-2xy+y^2&=6 \end{aligned}\right\}$$

18. $$\left.\begin{aligned} 2x^2-2xy+3y^2&=18\\ 3x^2-2y^2&=19 \end{aligned}\right\}$$

19. $$\left.\begin{aligned} x^2+3xy&=40\\ 4y^2+xy&=\ \ 9 \end{aligned}\right\}$$

20. $$\left.\begin{aligned} 2x^2+y^2&=\ \ 9\\ 5x^2+6y^2&=26 \end{aligned}\right\}$$

In Examples 20, 21, and 22, first eliminate y^2 or x^2.

21. $$\left.\begin{aligned} 4x^2-3y^2&=-11\\ 11x^2+5y^2&=\ \ 301 \end{aligned}\right\}$$

22. $$\left.\begin{aligned} 20x^2-16y^2&=179\\ 5x^2-336y^2&=\ \ 24 \end{aligned}\right\}$$

274. Systems in which the unknown letters are similarly involved may be solved as below:

EXAMPLE 1. Solve
$$\left.\begin{aligned} x^2+y^2&=74, && (1)\\ xy&=35, && (2) \end{aligned}\right\}\ (a)$$

Multiply (2) by 2, $$2xy=70. \qquad (3)$$

Add (3) to (1), $$x^2+2xy+y^2=144.$$

$$\left.\begin{aligned} \therefore\ x+y&=12, \text{ or } -12. && (4)\\ x-y&=2, \text{ or } -2. && (5) \end{aligned}\right\}\ (b)$$
Subtract (3) from (1),

System (b) is equivalent to the four systems

$$\left.\begin{aligned} x+y&=12,\\ x-y&=2. \end{aligned}\right\}\ \left.\begin{aligned} x+y&=12,\\ x-y&=-2. \end{aligned}\right\}\ \left.\begin{aligned} x+y&=-12,\\ x-y&=2. \end{aligned}\right\}\ \left.\begin{aligned} x+y&=-12,\\ x-y&=-2. \end{aligned}\right\}\ (c)$$

The solutions of systems (c) are 7, 5; 5, 7; -5, -7; -7, -5.

System (a) could be solved by the method of § 273.

Example 2. Solve

$$\left.\begin{aligned} x^2 - xy + y^2 &= 49, & (1)\\ x + y &= 13. & (2) \end{aligned}\right\} (a)$$

Square (2), $x^2 + 2xy + y^2 = 169.$ (3)

Subtract (1) from (3), $3xy = 120,$

or $xy = 40.$ (4)

Subtract (4) from (1), $x - y = 3, \text{ or } -3.$ (5)

Now (2) and (5) form a system equivalent to the two systems

$$\left.\begin{aligned} x + y &= 13,\\ x - y &= 3. \end{aligned}\right\} \qquad \left.\begin{aligned} x + y &= 13,\\ x - y &= -3. \end{aligned}\right\} \qquad (b)$$

The solutions of systems (*b*) are 8, 5, and 5, 8.

By substitution we find that these are the two solutions of (*a*). We cannot prove directly, as in previous solutions, that system (*a*) is equivalent to systems (*b*); for in squaring (2) to obtain (3), we introduced new solutions, which, as we find, were afterwards eliminated by taking (2) with (5) to form one final system.

By taking (1) with (5), we would obtain two systems with four solutions, two of which would not be solutions of (*a*).

Example 3. Solve

$$\left.\begin{aligned} x^4 + y^4 &= 82, & (1)\\ x - y &= 2. & (2) \end{aligned}\right\} (a)$$

Let $x = v + w,$ (3)

and $y = v - w.$ (4)

From (2), (3), and (4), $w = 1.$ (5)

From (1), (3), (4), and (5),

$$(v+1)^4 + (v-1)^4 = 82,$$

or $(v^2 + 10)(v^2 - 4) = 0.$

$$\therefore\ v = \pm 2, \text{ or } \pm\sqrt{-10}. \quad (6)$$

The equations (3) to (6) form system (*b*).

From (3), (5), and (6), $x = 3,\ -1,\ 1 \pm \sqrt{-10}.$

From (4), (5), and (6), $y = 1,\ -3,\ -1 \pm \sqrt{-10}.$ (*c*)

System (*a*) with (3) and (4) forms a system equivalent to (*b*), which is equivalent to (*c*) with (5) and (6).

Hence the four solutions of (*a*) are given in (*c*).

275. *In general, system* (*a*) *is equivalent to the two systems* (*b*) *and* (*c*).

$$\left.\begin{aligned} AB&=A'B', \quad (1)\\ B&=B'. \quad (2)\end{aligned}\right\}(a) \qquad \left.\begin{aligned} A&=A',\\ B&=B'.\end{aligned}\right\}(b) \qquad \left.\begin{aligned} B&=0,\\ B'&=0.\end{aligned}\right\}(c)$$

Substituting B for B' in (1), we obtain the system

$$\left.\begin{aligned} B(A-A')&=0,\\ B&=B'.\end{aligned}\right\}(d)$$

By § 204, system (*a*) is equivalent to system (*d*), and system (*d*) is evidently equivalent to (*b*) and (*c*); hence (*a*) is equivalent to (*b*) and (*c*).

If in any given system B and B' cannot each be zero, system (*c*) will be impossible, and system (*a*) will then be equivalent to (*b*).

It is important to note how systems (*b*) and (*c*) may be obtained from system (*a*).

Dividing equation (1) by (2), we obtain the first equation of (*b*); its second equation is equation (2).

System (*c*) is obtained by putting each member of equation (2) equal to zero.

EXAMPLE 1. Solve

$$\left.\begin{aligned} x^4+x^2y^2+y^4&=7371, \quad (1)\\ x^2-xy+y^2&=63. \quad (2)\end{aligned}\right\}(a)$$

Divide (1) by (2), $x^2+xy+y^2=117.$ (3)

Add (2) and (3),

Subtract (2) from (3),

$$\left.\begin{aligned} x^2+y^2&=90. \quad (4)\\ 2xy&=54. \quad (5)\end{aligned}\right\}(b)$$

System (*b*) can be solved by the method of § 274.

Here B', or 63, cannot equal zero; hence the system, $B=0$, $B'=0$, is impossible; therefore (*a*) is equivalent to the system (2) and (3), or to system (*b*).

Example 2. Solve

$$\left.\begin{aligned} x^3+y^3&=637, &(1)\\ x+y&=13. &(2)\end{aligned}\right\}(a)$$

Divide (1) by (2), $x^2-xy+y^2=49.$ (3)

Equations (2) and (3) form a system equivalent to (*a*).

Example 3. Solve

$$\left.\begin{aligned} x^2y+xy^2&=30, &(1)\\ x+y&=5. &(2)\end{aligned}\right\}(a)$$

Divide (1) by (2), $xy=6.$ (3)

Equations (2) and (3) form a system equivalent to (*a*).

Exercise 90.

Solve

1. $\left.\begin{aligned} x^2+y^2&=89\\ xy&=40\end{aligned}\right\}$

2. $\left.\begin{aligned} x^2+y^2&=170\\ xy&=13\end{aligned}\right\}$

3. $\left.\begin{aligned} x^2+y^2&=65\\ xy&=28\end{aligned}\right\}$

4. $\left.\begin{aligned} x^2+xy+y^2&=67\\ x^2-xy+y^2&=39\end{aligned}\right\}$

5. $\left.\begin{aligned} x^2+1+y^2&=3xy\\ 3x^2-xy+3y^2&=13\end{aligned}\right\}$

6. $\left.\begin{aligned} x^2-xy+y^2&=76\\ x+y&=14\end{aligned}\right\}$

7. $\left.\begin{aligned} x^2+xy+y^2&=61\\ x+y&=9\end{aligned}\right\}$

8. $\left.\begin{aligned} x^2-4xy+y^2&=52\\ \tfrac{1}{10}(x-y)&=1.\end{aligned}\right\}$

9. Solve systems 1, 2, 4, 8, 12, and 14 in Exercise 87 by the method employed in solving the system in Example 2 of § 274.

10. $\left.\begin{aligned} x^3+y^3&=3473\\ x+y&=23\end{aligned}\right\}$

11. $\left.\begin{aligned} x^3-y^3&=218\\ x-y&=2\end{aligned}\right\}$

12. $\left.\begin{aligned} x^3-y^3&=988\\ x-y&=4\end{aligned}\right\}$

13. $\left.\begin{aligned} x^3-y^3&=2197\\ x-y&=13\end{aligned}\right\}$

14. $\left.\begin{aligned} x^4+x^2y^2+y^4&=2128\\ x^2+xy+y^2&=76\end{aligned}\right\}$

15. $\left.\begin{aligned} x^4+x^2y^2+y^4&=243\\ x^2-xy+y^2&=9\end{aligned}\right\}$

16. $$\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 91 \\ x^2 + xy + y^2 &= 13 \end{aligned}\right\}$$

17. $$\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 2923 \\ x^2 - xy + y^2 &= 37 \end{aligned}\right\}$$

18. $$\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 7371 \\ x^2 - xy + y^2 &= 63 \end{aligned}\right\}$$

19. $$\left.\begin{aligned} x^3 - y^3 &= 56 \\ x^2 + xy + y^2 &= 28 \end{aligned}\right\}$$

20. $$\left.\begin{aligned} x^3 + y^3 &= 126 \\ x^2 - xy + y^2 &= 21 \end{aligned}\right\}$$

21. $$\left.\begin{aligned} \frac{2}{x} + \frac{1}{y} &= 1 \qquad (1) \\ \frac{2}{x^2} + \frac{3}{xy} + \frac{1}{y^2} &= 5 \qquad (2) \end{aligned}\right\}$$

Let $v = 1 \div x$, (3)

and $w = 1 \div y$. (4)

From (1), (3), and (4), $2v + w = 1$. (5)

From (2), (3), and (4), $2v^2 + 3vw + w^2 = 5$. (6)

$$\left.\begin{aligned} 2v + w &= 1 \qquad (5) \\ 2v^2 + 3vw + w^2 &= 5 \qquad (6) \end{aligned}\right\} (b)$$

The only solution of system (b) is -4, 9; hence the only solution of the given system is $-\frac{1}{4}$, $\frac{1}{9}$.

22. $$\left.\begin{aligned} 1 \div x + 1 \div y &= 2 \\ 1 \div x^2 + 1 \div y^2 &= 20 \end{aligned}\right\}$$

23. $$\left.\begin{aligned} \frac{1}{x^2} - \frac{1}{4y^2} &= 3 \\ \frac{1}{x^2} - \frac{1}{xy} + \frac{1}{4y^2} &= 9 \end{aligned}\right\}$$

24. $$\left.\begin{aligned} \frac{3}{x^2} - \frac{1}{y^2} &= 1 \\ \frac{5}{x^2} - \frac{1}{xy} + \frac{2}{y^2} &= 3 \end{aligned}\right\}$$

25. $$\left.\begin{aligned} x + y &= 1 \\ x^2y^2 + 13xy + 12 &= 0 \end{aligned}\right\}$$

The system in Example 25 is equivalent to the two systems

$$\left.\begin{aligned} x + y &= 1 \\ xy &= 12 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x + y &= 1 \\ xy &= 1 \end{aligned}\right\}$$

In the next two examples make the coefficients of y equal in the two equations and subtract.

26. $$\left.\begin{aligned} 2x^2 - xy + y^2 &= 2y \\ 2x^2 + 4xy &= 5y \end{aligned}\right\}$$

27. $$\left.\begin{aligned} x^3 + 1 &= 9y \\ x^2 + x &= 6y \end{aligned}\right\}$$

28. $$\left.\begin{aligned} x+y&=5 \\ 4xy&=12-x^2y^2 \end{aligned}\right\}$$

29. $$\left.\begin{aligned} x^2y+xy^2&=180 \\ x^2y^2&=400 \end{aligned}\right\}$$

30. $$\left.\begin{aligned} x+y-\sqrt{xy}&=7 \\ x^2+y^2+xy&=133 \end{aligned}\right\}$$

31. $$\left.\begin{aligned} 5x^2-5y^2&=x+y \\ 3x^2-3y^2&=x-y \end{aligned}\right\}$$

32. $$\left.\begin{aligned} x+y&=4 \\ x^4+y^4&=82 \end{aligned}\right\}$$

33. $$\left.\begin{aligned} x^4+y^4&=272 \\ x-y&=2 \end{aligned}\right\}$$

34. $$\left.\begin{aligned} \frac{x^2}{y}+\frac{y^2}{x}&=\frac{9}{2} \\ x+y&=3 \end{aligned}\right\}$$

35. $$\left.\begin{aligned} \frac{x+y}{x-y}+\frac{x-y}{x+y}&=\frac{5}{2} \\ x^2+y^2&=20 \end{aligned}\right\}$$

Exercise 91.

1. The difference of two numbers is 7, and the sum of their squares is 169. Find the numbers.

2. The sum of the squares of two numbers is 130, and the difference of their squares is 32. Find the numbers.

3. The sum of two numbers is 39, and the sum of their cubes is 17,199. Find the numbers.

4. A person bought some fine sheep for $360, and found that if he had bought 6 more for the same money, he would have paid $5 less for each. How many did he buy, and what was the price of each?

5. If the length and breadth of a rectangle were each increased by 1 yard, the area would be 48 square yards; if they were each diminished by 1 yard, the area would be 24 square yards. Find the length and breadth.

6. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two is $1\frac{5}{12}$; if the numerators were interchanged, the sum of the fractions would be $1\frac{1}{2}$. Find the fractions.

7. For a journey of 108 miles, 6 hours less would have sufficed, had the traveller gone 3 miles an hour faster. At what rate did he travel?

8. The hypothenuse of a right-angled triangle is 20 feet, and its area is 96 square feet. Find the length of the other two sides.

9. A number is divided into two parts such that the sum of the first and the square of the second is twice the sum of the second and the square of the first; and the sum of the number and the first part is 4 more than twice the second. Find the number.

10. The small wheel of a bicycle makes 135 revolutions more than the large wheel in a distance of 260 yards; if the circumference of each were one foot more, the small wheel would make 27 revolutions more than the large wheel in a distance of 70 yards. Find the circumference of each wheel.

11. A man bought 6 ducks and 2 turkeys for $15. For $14 he could buy 4 more ducks than he could turkeys for $9. Find the price of each.

12. The sum of the cubes of two numbers is 407, and the sum of their squares exceeds their product by 37. Find the numbers.

13. A rectangular field contains 160 square rods. If its length be increased by 4 rods, and its breadth by 3 rods, its area will be increased by 100 square rods. Find the length and breadth of the field.

14. A man rows down stream 12 miles in 4 hours' less time than it takes him to return. Should he row at twice his ordinary rate, his rate down stream would be 10 miles an hour. Find his rate in still water, and the rate of the stream.

15. The sum of two numbers is 7, and the sum of their fourth powers is 641. Find the numbers.

16. A gentleman left $210 to 3 servants to be divided in continued proportion, so that the first should have $90 more than the last. Find the legacy of each.

17. From a sheet of paper 14 inches long, a border of uniform width is cut away all round it, and the area is thereby reduced $\frac{5}{8}$; but had the sheet been 3 inches narrower, and a border of the same width had been cut away, the area would have been reduced $\frac{5}{7}$. Find the breadth of the paper, and the width of the border cut away.

18. A and B set out from the same place, and travel in the same direction at uniform rates. B starts 5 hours after A, and overtakes him after travelling 100 miles. Had their rates of travelling been a mile per hour less, B would have overtaken A after travelling 60 miles. Find their rates.

19. A man has to travel a certain distance, and, when he has travelled 40 miles, he increases his speed 2 miles per hour. If he had travelled with his increased speed during the whole journey, he would have arrived 40 minutes earlier; but if he had continued at his original speed, he would have arrived 20 minutes later. Find the whole distance he had to travel, and his original speed.

20. A cubical tank contains 512 cubic feet of water. It is required to enlarge the tank, the depth remaining the same, so that it shall contain 7 times as much water as before, subject to the condition that the length added to one side of the base shall be 4 times that added to the other. Find the sides of the new rectangular base.

CHAPTER XXIII.

THEORY OF LIMITS, VARIATION.

276. A Variable is a quantity that is, or is conceived to be, continually changing in value. Variable *numbers* are usually represented by the final letters of the alphabet, as x, y, z.

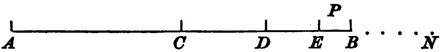
Thus the time since any past event is a variable; so, also, is the height of an ascending or a descending balloon. The amount of water in a cistern that is being filled by a continuous stream is a variable; and the number that measures this amount is a variable number.

A **Constant** is a quantity whose value is fixed or invariable. Constant *numbers* are usually represented by figures, or the first letters of the alphabet, as 4, 7, a, b, c.

Thus the time between any two past events is a constant; and the number that measures this time is a constant number.

277. Limit of a Variable. When, according to its law of change, a variable approaches indefinitely near a constant, but can never reach it, the constant is called the **Limit** of the variable. A variable may be always less, always greater, or alternately less and greater, than its limit.

Thus suppose a point P starting from A moves $\frac{1}{2}$ the distance from A to B, or to C, the first second; $\frac{1}{2}$ the remaining distance, or to D, the next second; $\frac{1}{2}$ the remaining distance, or to E, the next second; and so on. The point P will approach indefinitely



near to B, but can never reach it. Hence the variable distance AP will approach the constant distance AB as its limit; the variable distance NP will approach the constant distance NB as its limit; and the distance BP will approach zero as its limit.

The variable AP will always be less than its limit AB; and the variable NP will always be greater than its limit NB.

Some variables change according to such laws as to approach limits, others do not.

The theory of limits applies only to such variables as approach limits.

278. Notation. The sign $\doteq$ denotes "approaches as a limit"; thus, $x \doteq a$ is read 'x approaches a as its limit.'

The limit of x is often written briefly $\text{lt}(x)$.

279. *The difference between a variable and its limit is a variable whose limit is zero;* that is, if $\text{lt}(x) = a$, then $\text{lt}(a - x) = 0$.

Conversely, *if the difference between a variable and a constant is a variable whose limit is zero, the constant is the limit of the variable.*

Both of these propositions follow directly from the definition of the limit of a variable.

280. *A variable cannot approach two unequal limits at the same time.*

For in approaching the more remote of two unequal constants, the variable would evidently reach a value intermediate between the two constants, and thereafter, while it approached one of them as a limit, it would recede from the other, which therefore could not be a limit.

281. *The limit of the sum of a constant and a variable is the sum of the constant and the limit of the variable;* that is, $\text{lt}(c + x) = c + \text{lt}(x)$.

For let $\text{lt}\,(x) = a$; then, by definition, x approaches indefinitely near to a, but cannot reach it; but $c + x$ evidently approaches just as near to $c + a$ as x does to a; hence, by definition, $\text{lt}\,(c + x) = c + a = c + \text{lt}\,(x)$.

282. *If* $\text{lt}(v) = 0$, *then* $\text{lt}(cv) = 0$.

For let k denote any constant number, however small; then, since v can be made as small as we please, we may make $v < k \div c$. Multiplying this inequality by c, we have $cv < k$; that is, cv can be made as small as we please. But cv cannot be made zero since v cannot; hence, by definition, $\text{lt}\,(cv) = 0$.

283. *If two variables are always equal, and one approaches a limit, the other approaches the same limit;* that is, if $y = x$, and $x \doteq a$, then $y \doteq a$.

For if $x \doteq a$ and $y = x$, then y must approach indefinitely near to a, but cannot reach it; that is, $y \doteq a$.

284. *If two variables are always equal, and each approaches a limit, their limits are equal;* that is, if $y = x$, and $x \doteq a$, and $y \doteq b$, then $b = a$.

For if $y = x$, and $x \doteq a$, then, by § 283, $y \doteq a$. But $y \doteq b$, whence $b = a$; since, by § 280, y cannot approach two unequal limits at the same time.

285. *The limit of the product of a constant and a variable is the product of the constant and the limit of the variable;* that is, $\text{lt}(cx) = c\,\text{lt}(x)$.

Let $\quad \text{lt}(x) = a,$

and $\quad v = a - x; \quad (1)$

then $\quad \text{lt}(v) = \text{lt}(a - x) = 0. \quad$ §§ 279, 284

From (1), $\quad cx = ca - cv.$

$$\therefore \text{lt}(cx) = \text{lt}(ca - cv) \qquad \S\ 284$$
$$= ca - \text{lt}(cv) \qquad \S\ 281$$
$$= ca = c\,\text{lt}(x). \qquad \S\ 282$$

286. *The limit of the variable sum of two or more variables is the sum of their limits;* that is,

$$\text{lt}(x + y + z + \cdots) = \text{lt}(x) + \text{lt}(y) + \text{lt}(z) + \cdots$$

For let $\text{lt}(x) = a,\ \text{lt}(y) = b,$

and $v = a - x,\ w = b - y;$

then $x + y = (a + b) - (v + w).$

Hence
$$\text{lt}(x + y) = \text{lt}[(a + b) - (v + w)]$$
$$= a + b - \text{lt}(v + w) \qquad \S\ 281$$
$$= a + b = \text{lt}(x) + \text{lt}(y). \qquad (1)$$

Hence
$$\text{lt}(x + y + z + u)$$
$$= \text{lt}(x + y) + \text{lt}(z + u) \qquad \text{by } (1)$$
$$= \text{lt}(x) + \text{lt}(y) + \text{lt}(z) + \text{lt}(u).$$

287. *The limit of the variable product of two or more variables is the product of their limits;* that is,

$$\text{lt}(xyzu) = \text{lt}(x) \cdot \text{lt}(y) \cdot \text{lt}(z) \cdot \text{lt}(u).$$

For let $\text{lt}(x) = a,\ \text{lt}(y) = b,$

and $v = a - x,\ w = b - y;$

then $x = a - v,\ y = b - w.$

Hence $xy = ab - aw - bv + vw.$

$$\therefore \text{lt}(xy) = \text{lt}(ab - aw - bv + vw)$$
$$= ab + \text{lt}(-aw - bv + vw) \qquad \S\ 281$$
$$= ab - \text{lt}(aw) - \text{lt}(bv) + \text{lt}(vw) \qquad \S\ 286$$
$$= ab = \text{lt}(x) \cdot \text{lt}(y). \qquad (1)$$

Hence
$$\text{lt}(xyzu) = \text{lt}(xy) \cdot \text{lt}(zu) \qquad \text{by } (1)$$
$$= \text{lt}(x) \cdot \text{lt}(y) \cdot \text{lt}(z) \cdot \text{lt}(u).$$

288. *The limit of the variable quotient of two variables is the quotient of their limits;* that is,

$$\operatorname{lt}(x \div y) = \operatorname{lt}(x) \div \operatorname{lt}(y).$$

For let $z = x \div y$, or $x = yz$;

then $\operatorname{lt}(z) = \operatorname{lt}(x \div y)$, (1)

and $\operatorname{lt}(x) = \operatorname{lt}(yz) = \operatorname{lt}(y)\operatorname{lt}(z)$. (2)

From (2), $\operatorname{lt}(z) = \operatorname{lt}(x) \div \operatorname{lt}(y)$. (3)

From (1) and (3),

$$\operatorname{lt}(x \div y) = \operatorname{lt}(x) \div \operatorname{lt}(y).$$

REMARK. The proof fails, and the theorem is not true, when $\operatorname{lt}(y)$, or the limit of the divisor, is zero; for then we cannot divide the members of (2) by $\operatorname{lt}(y)$ to obtain (3).

EXAMPLE. Find $\operatorname{lt}\left(\frac{xyz}{cvw}\right)$.

$$\operatorname{lt}\left(\frac{xyz}{cvw}\right) = \frac{\operatorname{lt}(xyz)}{\operatorname{lt}(cvw)} \qquad \S\ 288$$

$$= \frac{\operatorname{lt}(x) \cdot \operatorname{lt}(y) \cdot \operatorname{lt}(z)}{c \cdot \operatorname{lt}(v) \cdot \operatorname{lt}(w)}. \qquad \S\S\ 287, 285$$

289. *When the product, quotient, or sum of two or more variables is equal to a constant, the product, quotient, or sum of their limits is equal to the same constant.*

(i.) Let $xy = m$; then $xyz = mz$.

$\therefore \operatorname{lt}(xyz) = \operatorname{lt}(mz)$. § 283

$\therefore \operatorname{lt}(x) \cdot \operatorname{lt}(y) \cdot \operatorname{lt} \cdot (z) = m \cdot \operatorname{lt}(z)$.

Divide by $\operatorname{lt}(z)$, $\operatorname{lt}(x) \cdot \operatorname{lt}(y) = m$.

The members of $xy = m$ are multiplied by the variable z to make them variables, so that we may apply § 284.

(ii.) Let $x \div y = m$; then $x = my$.

$\therefore \operatorname{lt}(x) = \operatorname{lt}(my) = m \cdot \operatorname{lt}(y)$.

$\therefore \operatorname{lt}(x) \div \operatorname{lt}(y) = m$.

[illegible]

[illegible] [illegible]

[illegible]

When [illegible]

is [illegible]

Infinite. [illegible]

Hence when [illegible]

and when [illegible]

Exercise [illegible]

1. Prove [illegible]

2. Prove $\text{lt}(m \div x^n) = m \div$ [illegible]

Let $m \div x^n = z$; then $m =$ [illegible]

Hence $\text{lt}(m \div x^n) = \text{lt}(z)$; and m [illegible]

$\therefore \text{lt}(m \div x^n) = m \div [\text{lt}(x)]^n.$

If $\text{lt}(x) = a$, $\text{lt}(y) = b$, $\text{lt}(z)$ [illegible]

3. $\text{lt}(xy + mxz)$.

4. [illegible]

5. $\text{lt}\left(\dfrac{x+my+nv}{xz-mz-ny}\right)$.

6. $\text{lt}\left(\dfrac{xv}{z}+\dfrac{mz^2}{ny}\right)$.

7. Find $\text{lt}\left(\dfrac{4x^3-3x^2+5}{7x^3+4x-8}\right)$, if $x=\infty$.

Dividing both dividend and divisor by x^3, we obtain

$$\frac{4x^3-3x^2+5}{7x^3+4x-8}=\frac{4-\dfrac{3}{x}+\dfrac{5}{x^3}}{7+\dfrac{4}{x^2}-\dfrac{8}{x^3}}.$$

$$\therefore\ lt\left(\frac{4x^3-3x^2+5}{7x^3+4x-8}\right)=\text{lt}\left(\frac{4-\dfrac{3}{x}+\dfrac{5}{x^3}}{7+\dfrac{4}{x^2}-\dfrac{8}{x^3}}\right) \qquad \S\ 284$$

$$=\frac{\text{lt}\left(4-\dfrac{3}{x}+\dfrac{5}{x^3}\right)}{\text{lt}\left(7+\dfrac{4}{x^2}-\dfrac{8}{x^3}\right)}=\frac{4}{7}.$$

Find the limit of each of the following expressions, when $x=\infty$:

8. $\dfrac{7x^2-3x}{5x^2+9}$.

10. $\dfrac{(3x-4)(5x+4)}{9x^2+8x-11}$.

9. $\dfrac{ax^3-bx+e}{mx^3+cx^2+nx}$.

11. $\dfrac{(3+2x^2)(2x-7)}{(5x^2+7)(7+9x)}$.

LAWS OF INCOMMENSURABLE NUMBERS.

291. The Algebraic laws, already proved for commensurable numbers, are, by the theory of limits, easily proved for incommensurable numbers.

292. *To prove the fundamental laws.*

Let a, b, c, be any incommensurable constant numbers, and let x, y, z, be commensurable variable numbers such that $\text{lt}(x) = a$, $\text{lt}(y) = b$, $\text{lt}(z) = c$;

then $$xy = yx.$$

$$\therefore\ \text{lt}(x)\text{lt}(y) = \text{lt}(y)\text{lt}(x),$$

or $$a \cdot b = b \cdot a. \qquad \text{Law (A}')$$

Again, $$(x + y)z = xz + yz.$$

$$\therefore\ [\text{lt}(x) + \text{lt}(y)]\text{lt}(z) = \text{lt}(x)\text{lt}(z) + \text{lt}(y)\text{lt}(z),$$

or $$(a + b)c = ac + bc. \qquad \text{Law (C)}$$

Similarly the other fundamental laws may be proved for incommensurable numbers.

293. *If* x *is commensurable and* $x \doteq 0$, *then* $a^x \doteq 1$.

To show the truth of this principle, let us consider the exponential expression 64^x. Giving to x successive values and finding the corresponding values of 64^x, we obtain the results given below:

when $x = 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{6},\ \frac{1}{12},\ \frac{1}{24},\ \frac{1}{48},\ \frac{1}{96},\ \frac{1}{192},\ \frac{1}{384},\ \cdots,$

$64^x = 64,\ 8,\ 4,\ 2,\ 1.4,\ 1.19,\ 1.09,\ 1.04,\ 1.019,\ 1.009,\ \cdots.$

From this table of values we see that as $x \doteq 0$, $64^x \doteq 1$.

From this principle it follows also that $a^0 = 1$.

294. *If* x *is commensurable,*

and $$\text{lt}(x) = m,$$

then $$\text{lt}(a^x) = a^m.$$

Let $$x < m \text{ and } z > m,$$

x and z both being commensurable variables; then, if $a > 1$, we assume as axiomatic that

$$a^x < a^m \text{ and } a^z > a^m. \qquad (1)$$

Let $\text{lt}(z) = m$; then $\text{lt}(z - x) = \text{lt}(z) - \text{lt}(x) = 0$; hence, by § 293,

$$\text{lt}(a^{z-x}) = 1. \tag{2}$$

Now $$a^z - a^x = a^x(a^{z-x} - 1).$$

$$\begin{aligned} \therefore \text{lt}(a^z - a^x) &= \text{lt}(a^x)\text{lt}(a^{z-x} - 1) && \text{§§ 284, 287} \\ &= \text{lt}(a^x)[\text{lt}(a^{z-x}) - 1] && \text{§ 281} \\ &= \text{lt}(a^x)(1 - 1) = 0. && \text{By (2)} \end{aligned}$$

From (1) $$a^z - a^x > a^m - a^x;$$

hence, as $\text{lt}(a^z - a^x) = 0$, $\text{lt}(a^m - a^x) = 0$,

or $$a^m = \text{lt}(a^x).$$

295. *To prove the laws of exponents.*

Let m and n be any incommensurable constant numbers, and let x and y be commensurable variable numbers such that $\text{lt}(x) = m$, $\text{lt}(y) = n$.

(i.) $$\text{lt}(a^x a^y) = \text{lt}(a^x) \cdot \text{lt}(a^y) = a^m a^n. \tag{1}$$

But $$a^x a^y = a^{x+y}.$$

$$\therefore \text{lt}(a^x a^y) = \text{lt}(a^{x+y}) = a^{m+n}. \tag{2}$$

From (1) and (2), $a^m a^n = a^{m+n}$. Law (I.)

(ii.) $$\text{lt}\left(\frac{a^x}{a^y}\right) = \frac{\text{lt}(a^x)}{\text{lt}(a^y)} = \frac{a^m}{a^n}. \tag{3}$$

But $$a^x \div a^y = a^{x-y}.$$

$$\therefore \text{lt}(a^x \div a^y) = \text{lt}(a^{x-y}) = a^{m-n}. \tag{4}$$

From (3) and (4), $\dfrac{a^m}{a^n} = a^{m-n}$. (Law II.)

Similarly the other laws of exponents may be proved for incommensurable exponents.

VARIATION.

296. Two variables may, in an unlimited number of ways, be so related that one will depend upon the other for its value.

Thus the variable distance which a boy runs will depend upon the time and rate. The length of an elastic cord depends upon its tension. The value of $4x$ or $3x^2$ depends upon the value of the variable x.

297. One variable is said to *vary as* another when their ratio is constant. The sign $\propto$, called the *sign of variation*, is read '*varies as*.' When placed between two expressions, it denotes that their ratio is constant, or that the first equals the second into a constant.

Thus $y \propto x$, read 'y varies as x,' denotes that $y = mx$, m denoting some constant.

Hence, if the altitude of a rectangle is *constant*,

the area $\propto$ the base;

for the area = the base into the *constant* altitude.

298. If $y \propto x$, and if y', x' and y'', x'' be any two sets of corresponding values of y and x, then

$$y' = mx', \text{ and } y'' = mx''.$$

$$\therefore\ y' : x' = y'' : x''.$$

Conversely, if $y' : x' = y'' : x''$, then $y \propto x$.

299. One variable is said to *vary inversely as* another when the first varies as the *reciprocal* of the second.

Thus y varies *inversely* as x, if $y \propto 1 \div x$; that is, if $y = m(1 \div x)$, or $xy = m$.

One variable is said to vary as two others *jointly*, when the first varies as the product of the other two.

Thus z varies as x and y *jointly*, if $z \propto xy$, that is if $z = mxy$.

One variable is said to *vary directly* as a second and *inversely* as a third, when the first varies as the quotient of the second by the third.

Thus z varies directly as x and inversely as y, if $z \propto x \div y$, that is, if $z = m(x \div y)$.

In each of the cases of variation given above, the value of the constant m can be found when any set of corresponding values of x, y, and z is given.

For example, if $z \propto x \div y$, and $z = 6$ when $x = 2$ and $y = 9$, we have $z = m(x \div y)$. $\therefore\ 6 = m(2 \div 9)$.

Hence $m = 27$. $\therefore\ z = 27(x \div y)$.

300. The simplest method of treating *variations* is to convert them into *equations*.

Example. If $u \propto y$ and $y \propto x$, prove that $u \propto x$.

Since $u \propto y$ and $y \propto x$, by § 297 we have

$u = ay$ and $y = bx$, where a and b are constants.

$\therefore\ u = abx.$ $\therefore\ u \propto x.$ § 297

301. *If* u ∝ x *when* y *is constant, and* u ∝ y *when* x *is constant, then* u ∝ xy *when* x *and* y *both vary.*

Let x', y', u', be one set and x'', y'', u'' another set of corresponding values of x, y, z, when all change together.

Let x change from x' to x'', y remaining constant, and suppose that in consequence u changes from u' to u_1; then since $u \propto x$ when y is constant, we have by § 298

$$u' : x' = u_1 : x''. \qquad (1)$$

Now let y change from y' to y'', x remaining constant; then u will change from u_1 to u''; hence as $u \propto y$ when x is constant, we have

$$u_1 : y' = u'' : y''. \qquad (2)$$

Multiplying (1) by (2), and dividing the antecedents by u_1, we have

$$u' : x'y' = u'' : x''y''.$$

Hence $u \propto xy$.

Similarly it may be proved that, if u varies as each one of the three variables x, y, z when the rest are constant, then $u \propto xyz$ when they all change; and so on.

The following examples illustrate the proposition given above.

Let W denote the amount of work done by M men in D days; then

$W \propto M$, when D is constant,

and $W \propto D$, when M is constant;

hence $W \propto DM$, when D and M both change.

Again, let A denote the area of a triangle, B its base, and H its altitude; then

$A \propto B$, when H is constant,

and $A \propto H$, when B is constant;

hence $A \propto BH$, when B and H both change.

Example. The volume, V, of a pyramid varies jointly as its height, H, and the area of its base, B; and when the area of the base is 60 square feet and the height 14 feet, the volume is 280 cubic feet. Find the area of the base of a pyramid whose volume is 390 cubic feet, and whose height is 26 feet.

Since $V \propto BH$, $V = mBH$, in which m is a constant.

Substituting the given values of V, B, H, we have

$$280 = m \times 60 \times 14, \text{ or } m = \tfrac{1}{3}.$$

$$\therefore\ V = \tfrac{1}{3}BH.$$

Hence when $V = 390$, and $A = 26$, we have

$$390 = \tfrac{1}{3}B \times 26.$$

$$\therefore\ B = 45, \text{ the number of sq. ft. in base.}$$

Exercise 93.

1. If x varies as y, and $y = 7$ when $x = 18$; find x when $y = 21$

2. If y varies inversely as x, and $y = 4$ when $x = 15$; find y when $x = 12$.

3. If x varies jointly as y and z, and $x = 6$ when $y = 3$, $z = 2$; find x when $y = 5$, $z = 7$.

4. If $x^2 \propto y$ and $z^2 \propto y$, then $xz \propto y$.

5. If $x \propto 1 \div y$, and $y = 4$ when $x = 15$; find y when $x = 6$.

6. If x varies as y directly and as z inversely, and $x = 10$ when $y = 15$, $z = 6$; find x when $y = 8$, $z = 2$.

7. If x varies as y directly and as z inversely, and $x = 14$ when $y = 10$, $z = 14$; find z when $x = 49$, $y = 45$.

8. If $x \propto 1 \div y$, and $y \propto 1 \div z$, prove $z \propto x$.

9. If $3x + 7y \propto 3x + 13y$, and when $x = 5$, $y = 3$; find the equation between x and y.

10. If the cube of x varies as the square of y, and if $x = 3$ when $y = 5$; find the equation between x and y.

11. If the area of a circle varies as the square of its radius, and if the area of a circle is 154 square feet when the radius is 7 feet; find the area of a circle whose radius is 10 feet 6 inches.

12. The velocity of a falling body varies as the time during which it has fallen from rest, and the velocity at the end of 2 seconds is 64. Find the velocity at the end of 5 seconds.

13. The volume of a sphere varies as the cube of its radius and the volume of a sphere whose radius is 1 foot is 4.188 cubic feet. Find the volume of a sphere whose radius is 3 feet.

14. The pressure of a gas varies jointly as its density and its absolute temperature; also when the density is 1 and the temperature 300, the pressure is 15. Find the pressure when the density is 3 and the temperature is 320.

15. The volume of a gas varies as the absolute temperature and inversely as the pressure. Also when the pressure is 15 and the temperature 280, the volume is 1 cubic foot. Find the volume when the pressure is 20 and the temperature 300.

16. The distance through which a heavy body will fall from rest varies as the square of the time, and a body will fall through 144 feet in 3 seconds. Find how far it will fall in 2 seconds.

17. The pressure of wind on a plane surface varies jointly as the area of the surface, and the square of the wind's velocity. The pressure on a square foot is 1 pound when the wind is moving at the rate of 15 miles per hour. Find the velocity of the wind when the pressure on a square yard is 16 pounds.

18. The volume of a right circular cone varies jointly as its height and the square of the radius of its base; and the volume of a cone 7 feet high with a base whose radius is 3 feet is 66 cubic feet. Find the volume of a cone 9 feet high with a base whose radius is 14 feet.

CHAPTER XXIV.

INDETERMINATE EQUATIONS AND SYSTEMS, $a \div 0$ AND $0 \div 0$, INEQUALITIES.

302. An **Impossible Equation** or **System of Equations*** is one that has *no finite* solution. The equations of an impossible system are said to be **Inconsistent.**

Thus $3x + 5 = 3x - 8$ is an impossible equation; for its equivalent equation $0 \cdot x = -13$ has no finite solution. Again, $2 - \sqrt{2x+8} + 2\sqrt{x+5} = 0$ is an impossible equation; for it has no solution (Example 3, § 268).

The system

$$\left.\begin{aligned} ax + by &= c, & (1)\\ 3ax + 3by &= 5c, & (2) \end{aligned}\right\}$$

is impossible; for multiplying (1) by 3 and subtracting the resulting equation from (2), we obtain the impossible equation

$$0 \cdot x + 0 \cdot y = 2c,$$

which shows that equations (1) and (2) are inconsistent.

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* An impossible equation or system of equations is, in general, but the limiting case of a more general equation or system, the solutions of which in the limit become infinity.

Thus, the equation $ax = b$ becomes impossible only when $a = 0$, and then its root $b \div a$ becomes $b \div 0$, or infinity.

It will be seen in § 306 that a system of linear equations becomes impossible only for a certain relation between the coefficients of its equations, which renders both x and y infinite.

Again, the general system

$$\left.\begin{aligned} a^2x^2 - b^2y^2 &= c^2,\\ ax - (b+e)y &= c, \end{aligned}\right\}$$

becomes the defective system (a), in § 303, only when $e = 0$.

The system

$$\left.\begin{aligned} x + y &= 9, & (1)\\ 2x + y &= 13, & (2)\\ x + 2y &= 16, & (3)\end{aligned}\right\}$$

is impossible; for solving (1) and (2), we obtain $x = 4$, $y = 5$; substituting these values in (3), we obtain not an identity, but the absurdity, $14 = 16$. Similarly the solution of any other two of these three equations will not satisfy the third.

Here the number of *independent* equations exceeds the number of unknown letters.

303. A **Defective System** is one that lacks one or more of the full number of solutions.

Thus, if in the system

$$\left.\begin{aligned} a^2x^2 - b^2y^2 &= c^2, & (1)\\ ax - by &= c, & (2)\end{aligned}\right\}\ (a)$$

we divide equation (1) by (2), we obtain

$$ax + by = c. \qquad (3)$$

Now (2) and (3) form a system equivalent to system (a); hence (a) has but one solution. But by § 271 such a system as (a) has in general two solutions. Hence system (a) is a *defective* system. Many of the systems in Exercise 89 are defective.

304. An **Indeterminate Equation** or **System of Equations** is one that admits of an infinite number of solutions. Hence a single equation containing two or more unknown letters is indeterminate (§ 194). Again, any system of equations that contains more *unknown* letters than *independent* equations is indeterminate.

Thus $3x - 2y = 4$ is an indeterminate equation.

Again, the system $\left.\begin{aligned} 3x + 4y + 5z &= 0,\\ x - y \quad - 2z &= 0,\end{aligned}\right\}$

is an indeterminate system; for finding the values of y and z in terms of x, we may give to x any value, and find the corresponding values of y and z.

Here the number of independent equations is *less* than the number of unknown letters.

The system

$$\left.\begin{aligned} 3x-4y &= 9, \\ 6x-8y &= 18, \end{aligned}\right\}$$

is indeterminate, for its equations are not independent, but equivalent.

Again, the system

$$\left.\begin{aligned} 2x+3y-z &= 15, &\quad (1)\\ 3x-y+2z &= 8, &\quad (2)\\ 5x+2y+z &= 23, &\quad (3) \end{aligned}\right\}$$

is indeterminate. No two of its equations are equivalent, but any one of them can be obtained from the other two; thus, by adding (1) and (2), we obtain (3). Hence the system contains but two independent equations.

305. Sometimes it is required to find the *positive integral* solutions of an indeterminate equation or system. The following examples will illustrate the simplest general method of finding such solutions.

Example 1. Solve $7x+12y=220$ in positive integers.

Dividing by 7, the smaller coefficient, expressing improper fractions as mixed numbers, and combining, we obtain

$$x+y+\frac{5y-3}{7}=31. \qquad (1)$$

Since x and y are integers, $31-x-y$ is an integer; hence the fraction in (1), or any integral multiple of it, equals an integer. Multiplying this fraction by such a number as to make the coefficient of y divisible by the denominator with remainder 1, which in this case is 3, we have

$$\frac{15y-9}{7}=2y-1+\frac{y-2}{7}=\text{an integer}.$$

Hence $\dfrac{y-2}{7}=$ an integer $=p$, suppose.

$$\therefore\ y=7p+2. \qquad (2)$$

From (1) and (2), $x=28-12p$. (3)

Since x and y are *positive* integers, from (2) it follows that $p > -1$, and from (3) it follows that $p < 3$; hence

$$p = 0, 1, 2. \qquad (4)$$

From (2), (3), and (4), we obtain the three solutions

$$x = 28, 16, 4;$$
$$y = 2, 9, 16.$$

Example 2. Solve in positive integers the system

$$\left.\begin{aligned} x + y + z &= 43, &(1)\\ 10x + 5y + 2z &= 229. &(2)\end{aligned}\right\}$$

Eliminate z, $8x + 3y = 143$,

or $$y + 2x + \frac{2x-2}{3} = 47. \qquad (3)$$

$$\therefore \frac{4x-4}{3} = x - 1 + \frac{x-1}{3} = \text{an integer.}$$

$$\therefore \frac{x-1}{3} = \text{an integer} = p, \text{ suppose.}$$

$$\therefore x = 3p + 1. \qquad (4)$$

From (3) and (4), $y = 45 - 8p$. (5)

From (1), (4), and (5), $z = 5p - 3$. (6)

From (6), $p > 0$; and from (5), $p < 6$; hence

$$p = 1, 2, 3, 4, 5.$$

Whence
$$x = 4, 7, 10, 13, 16;$$
$$y = 37, 29, 21, 13, 5;$$
$$z = 2, 7, 12, 17, 22.$$

Thus, the system has five positive integral solutions.

Exercise 94.

Solve in positive integers

1. $3x + 29y = 151$.
2. $3x + 8y = 103$.
3. $7x + 12y = 152$.
4. $13x + 7y = 408$.
5. $23x + 25y = 915$.
6. $13x + 11y = 414$.
7. $\left.\begin{aligned} 6x + 7y + 4z &= 122\\ 11x + 8y - 6z &= 145\end{aligned}\right\}$

8. $\left.\begin{aligned} 12x - 11y + 4z &= 22 \\ -4x + 5y + z &= 17 \end{aligned}\right\}$

9. $\left.\begin{aligned} 20x - 21y &= 38 \\ 3y + 4z &= 34 \end{aligned}\right\}$

10. $5x - 14y = 11.$

11. $\left.\begin{aligned} 13x + 11z &= 103 \\ 7z - 5y &= 4 \end{aligned}\right\}$

12. $14x - 11y = 29.$

13. A farmer buys horses at \$111 a head, cows at \$69, and spends \$2256; how many of each does he buy?

14. A drover buys sheep at \$4 a head, pigs at \$2, and oxen at \$17; if 40 animals cost him \$301, how many of each kind does he buy?

15. I have 27 coins, which are dollars, half-dollars, and dimes, and they amount to \$9.80; how many of each sort have I?

16. A drover buys sheep at \$3.50 a head, turkeys at \$1.33⅓, and hens at \$0.50. If 100 animals cost him \$100, how many of each does he buy?

306. The expressions $0 \div a$, $a \div 0$, $0 \div 0$.

When $x \doteq 0$, $x \div a \doteq 0$; hence the expression $0 \div a = 0$.

When $x \doteq 0$, $a \div x = \infty$; hence the expression $a \div 0$ represents that which transcends all number, or *absolute infinity*. As a solution of an equation, $a \div 0$ denotes that there is no number that will satisfy the given conditions.

As a quotient, the expression $0 \div 0$ denotes the number which multiplied by zero equals zero; but any number whatever multiplied by zero equals zero. Hence the expression $0 \div 0$ represents any number whatever; for this reason, it is often called the **Symbol of Indetermination.**

The following problem illustrates the meaning of the expressions, $a \div 0$ and $0 \div 0$.

Problem. A and B are travelling in the direction PR at the rates of a and b miles per hour. At 12 o'clock A is at P and B at Q, which is c miles to the right of P. Find when they are together.

Let distance measured to the right from P, and time reckoned after 12 o'clock, be regarded as positive.

Let $x =$ the number of hours from 12 o'clock to the time of their being together.

Then $$ax = bx + c. \qquad (1)$$

Hence $$x = \frac{c}{a-b}. \qquad (2)$$

Discussion. If c is not zero, and $a > b$, x will be positive; that is, A will overtake B at some time after 12 o'clock.

If c is not zero, and $a < b$, x will be negative; that is, A and B were together at some time *before* 12 o'clock.

If the problem were to find at what time *after* 12 o'clock A would overtake B, it would require a positive number for an answer, and would be impossible for the last hypothesis.

If $c = 0$, and $a > b$ or $a < b$, $x = 0$; that is, A and B are together at 12 o'clock, but not before or after that time.

If c is not zero, and $a = b$, $x = c \div 0$, or absolute infinity; that is, A and B are not together at, before, or after 12 o'clock, and the problem is impossible.

If $c = 0$, and $a = b$, $x = \frac{0}{0}$; that is, there is an infinite number of times when A and B are together, and the problem is indeterminate.

Each of the foregoing conclusions evidently agrees with the conditions which they interpret.

Example. By discussing its solution show that the system

$$\left.\begin{aligned} ax + by &= c, \\ a'x + b'y &= c' \end{aligned}\right\} \quad (a)$$

is (i.) indeterminate if $$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}; \qquad (1)$$

and (ii.) impossible if $$\frac{a}{a'} = \frac{b}{b'} \text{ not } = \frac{c}{c'}. \qquad (2)$$

By § 203 the solution of system (a) is

$$x = \frac{b'c - bc'}{ab' - a'b}, \quad y = \frac{ac' - a'c}{ab' - a'b} \qquad (3)$$

(i.) If relation (1) exists, then from (1) we have $ab' - a'b = 0$, $b'c - bc' = 0$, $ac' - a'c = 0$; hence the values of x and y in (3) each assume the form $\frac{0}{0}$, and the system has an infinite number of solutions; that is, it is indeterminate.

(ii.) If relation (2) exists, then $ab' - a'b = 0$. But neither $b'c - bc'$ nor $ac' - a'c$ is zero; hence x and y are each infinite, and the system is impossible.

It is evident that the equations in (a) are equivalent when (1) is satisfied, and inconsistent when (2) holds true.

307. The fraction $\frac{c}{a - b}$ assumes the form $\frac{0}{0}$ by reason of *two independent suppositions;* namely, $c = 0$ and $a = b$. In all such cases $0 \div 0$ indicates *indetermination.*

If, however, a fraction assumes the form $0 \div 0$ for a *single supposition,* the fraction will be found to have a *single definite* value for that supposition.

For example, the fraction $\frac{x^3 - 1}{x^2 - 1}$ assumes the form $\frac{0}{0}$, for the supposition that $x = 1$.

To find the value of this fraction for $x = 1$, we find its limit when $x \doteq 1$.

For any value of x except $x = 1$, we have

$$\frac{x^3 - 1}{x^2 - 1} = \frac{x^2 + x + 1}{x + 1}. \qquad (1)$$

Hence (1) holds true when $x \doteq 1$; therefore by § 284

$$\text{lt}\left(\frac{x^3 - 1}{x^2 - 1}\right) = \text{lt}\left(\frac{x^2 + x + 1}{x + 1}\right)$$

$$= \frac{\text{lt}(x^2 + x + 1)}{\text{lt}(x + 1)} = \frac{3}{2}.$$

Hence the value of the fraction for $x = 1$ is $\frac{3}{2}$.

We cannot apply the theory of limits directly to the given fraction, for the limit of its divisor is 0 when $x \doteq 1$.

INEQUALITIES.

308. An **Inequality** is the statement that one number is greater or less than another. $a > b > c$ indicates that $b < a$ and $> c$; $a \overline{\gtrless} b$ indicates that $a =$ or $> b$.

309. The following principles used in transforming inequalities will, upon a little reflection, become evident:

(i.) An inequality will still hold after both members have been

Increased or diminished by the same number;

Multiplied or divided by the same positive number;

Raised to any *odd* power, or to *any* power if both members are essentially positive.

(ii.) The sign of an inequality must be reversed after both members have been

Multiplied or divided by the same negative number;

Raised to the same even power, if both members are negative.

(iii.) If the same root be extracted of both members of an inequality, the sign must be reversed only when negative even roots are compared.

310. In establishing the relation of inequality between two symmetrical expressions, the following principle is very useful.

If a *and* b *are unequal and real,* $a^2 + b^2 > 2ab$.

For $(a - b)^2 > 0,$

or $$a^2 - 2ab + b^2 > 0; \qquad (1)$$

hence $$a^2 + b^2 > 2ab. \qquad (2)$$

Example 1. Prove that $\frac{1}{2}(x + y) > \sqrt{xy}$.

If in (2) we put $a^2 = x$ and $b^2 = y$, we obtain

$$x + y > 2\sqrt{xy}, \text{ or } \tfrac{1}{2}(x + y) > \sqrt{xy}.$$

Example 2. Show that $a^3 + b^3 > a^2b + ab^2$, if $a + b > 0$.

From (1), $a^2 - ab + b^2 > ab.$

Multiply by $a + b$, $a^3 + b^3 > a^2b + ab^2.$

Example 3. Simplify $\frac{1}{3}(5x - 7) > \frac{1}{5}(2 - 3x).$

Multiply by 15, $25x - 35 > 6 - 9x.$

Transpose, $34x > 41$, or $x > \frac{41}{34}.$

Example 4. Given

$$4x - 6 < 2x + 4, \qquad (1)$$

$$2x + 4 > 16 - 2x, \qquad (2)$$

to find the limiting values of x.

From (1), $2x < 10$, or $x < 5.$

From (2), $4x > 12$, or $x > 3.$

Hence the values of x lie between 3 and 5.

Exercise 95.

1. Simplify $6x > \frac{3}{2}x + 18.$

2. Simplify $\frac{7}{5}x - \frac{5}{8}x > \frac{1}{8}x - 3.$

3. Simplify $m - nx > p - qx.$

4. If $5x - 6 < 3x + 8$, and $2x + 1 < 3x - 3$, show that the values of x lie between 4 and 7.

5. If $3x - 2 > \frac{1}{2}x - \frac{4}{5}$, and $\frac{7}{6} - \frac{5}{4}x < 8 - 2x$, show that the values of x lie between $\frac{12}{25}$ and $\frac{82}{9}$.

6. Show that the sum of any fraction and its reciprocal is greater than 2, numerically.

The letters denoting unequal positive numbers, prove

7. $a^2 + b^2 + c^2 > ab + ac + bc$; $m^3 + 1 > m^2 + m.$

8. $\frac{a + b}{2} > \frac{2ab}{a + b}$; $\frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{b} + \frac{1}{a}.$

9. $a^3 + b^3 + c^3 > \frac{1}{2}(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2).$

10. If $a^2 + b^2 + c^2 = 1$, and $m^2 + n^2 + r^2 = 1$, show that $am + bn + cr < 1.$

CHAPTER XXV.

THE PROGRESSIONS.

311. A **Series** is a succession of numbers that follow each other according to some general law. The successive numbers are called the **Terms** of the series.

A series is said to be **Finite** or **Infinite,** according as the *number* of its terms is *finite* or *infinite.*

Of the unlimited number of different forms of series, we shall consider in this chapter only the Arithmetical, the Geometric, and the Harmonic, Progressions.

ARITHMETICAL PROGRESSION.

312. An **Arithmetical Progression** is a series in which the difference between any term and the preceding term is the same throughout the series. This difference is called the *Common Difference.* It may be either positive or negative.

Thus the series

$$2,\ 5,\ 8,\ 11,\ 14,\ 17,\ 20,\ 23,\cdots$$

and

$$7,\ 5,\ 3,\ 1,\ -1,\ -3,\ -5,\ -7,\cdots$$

are arithmetical progressions (A. P.). In the first series the common difference is 3; and in the second series it is -2. Thus if in the second series we add -2 to any term, we obtain the next term.

313. The *n*th Term. Let d denote the common difference, and a the first term; then by definition

$$\begin{aligned} &\text{the second term} &&= a + d, \\ &\text{the third term} &&= a + 2d, \\ &\text{the fifth term} &&= a + 4d, \\ \text{and}\quad &\text{the } n\text{th term} &&= a + (n-1)d. \qquad (1) \end{aligned}$$

Thus if the first term of an A. P. is 4, and the common difference is 5,

$$\begin{aligned} &\text{the ninth term} &&= 4 + (9 - 1)5 = 44, \\ \text{and}\quad &\text{the twenty-first term} &&= 4 + (21 - 1)5 = 104. \end{aligned}$$

Example. The fifty-fourth and fourth terms of an A. P. are respectively -61 and 64; find the twenty-seventh term.

$$\left.\begin{aligned} \text{Here}\quad -61 &= \text{the fifty-fourth term} = a + 53d, \\ \text{and}\quad 64 &= \text{the fourth term} = a + 3d. \end{aligned}\right\} \qquad (a)$$

Solving system (a), we find $a = 71\frac{1}{2}$, $d = -\frac{5}{2}$.

$$\therefore \text{ the twenty-seventh term} = a + 26d = 6\tfrac{1}{2}.$$

314. When three numbers, a, b, c, are in A. P., the middle term b is called the **Arithmetical Mean** of the other two terms a and c.

315. If a, b, c are in A. P., we have

$$b - a = c - b.$$

$$\therefore\ b = \tfrac{1}{2}(a + c).$$

That is, *the arithmetical mean of any two numbers is half their sum.*

316. All the terms between any two terms of an A. P. may be called the **Arithmetical Means** of the two terms.

Between any two given numbers, any number of arithmetical means may be inserted.

Example. Insert 9 arithmetical means between 50 and 80.

Since there are 9 arithmetical means, 80 must be the eleventh term, 50 being the first; hence we have

$$\text{the eleventh term} = 50 + 10d = 80.$$

Hence $d = 3$, and the required series is

$$50,\ 53,\ 56,\ 59,\ 62,\ 65,\ 68,\ 71,\ 74,\ 77,\ 80.$$

317. Sum of n Terms. Let l denote the nth term, and s the sum of n terms of the series; then

$$s = a + (a+d) + (a+2d) + \cdots + (l-d) + l,$$

or

$$s = l + (l-d) + (l-2d) + \cdots + (a+d) + a.$$

Adding the corresponding terms, we have

$$2s = (a+l) + (a+l) + (a+l) + \cdots \text{ to } n \text{ terms}$$
$$= n(a+l).$$
$$\therefore\ s = \tfrac{1}{2}n(a+l). \qquad (1)$$

From § 313, $l = a + (n-1)d. \qquad (2)$

$$\therefore\ s = \tfrac{1}{2}n\{2a + (n-1)d\}. \qquad (3)$$

If any three of the five numbers a, d, l, n, s be given, the other two may be found from formulas (1) and (2), or from (3) and (2).

Example 1. Find the sum of 20 terms of the A. P.

$$-5 - 1 + 3 + 7 + 11 + \cdots.$$

Here $a = -5,\ d = 4,\ n = 20.$

$$\therefore\ s = \tfrac{1}{2}n\{2a + (n-1)d\}$$
$$= 10\{-10 + 19 \times 4\}$$
$$= 660.$$

Example 2. Find the sum of the first n consecutive odd numbers, 1, 3, 5 $\cdots$.

Here $$a = 1,\ d = 2,\ n = n.$$
$$\begin{aligned}\therefore\ s &= \tfrac{1}{2} n \{2a + (n-1)d\}\\ &= \tfrac{1}{2} n \{2 + (n-1)2\}\\ &= n^2.\end{aligned}$$

Hence the sum of n consecutive odd numbers beginning with unity is n^2.

EXAMPLE 3. The first term of an A. P. is 6, and the sum of 25 terms is 25. Find the common difference.

Here $a = 6$, $s = 25$, $n = 25$; hence from (3) of § 317, we have
$$25 = \tfrac{1}{2} \times 25\{12 + 24d\}.$$
$$\therefore\ d = -\tfrac{5}{12}.$$

EXAMPLE 4. How many terms must be taken of the series 11, 12, 13, ⋯ to make 410?

Here $a = 11$, $d = 1$, $s = 410$; hence from (3) of § 317, we have
$$410 = \tfrac{1}{2} n\{22 + (n-1)\}. \tag{1}$$
$$\therefore\ n = 20, \text{ or } -41.$$

Since the number of terms must be an arithmetical whole number, the number of terms is 20.

EXAMPLE 5. How many terms must be taken of the series -16, -15, -14, ⋯ to make -100?

Here $a = -16$, $d = 1$, $s = -100$; hence we have
$$-100 = \tfrac{1}{2} n\{-32 + (n-1)\}.$$
$$\therefore\ n = 8, \text{ or } 25.$$

Hence the number of terms is 8 or 25. The sum of the 17 terms following the first 8 must therefore be zero. These 17 terms are -8, -7, $-$ ⋯, 7, 8, and their sum is evidently zero.

Exercise 96.

1. Find the twenty-seventh and forty-first terms in the series 5, 11, 17, ⋯.

2. Find the seventeenth and fifty-fourth terms in the series 10, $11\frac{1}{2}$, 13, ⋯.

3. Find the twentieth and thirteenth terms in the series $-3, -2, -1, \cdots$.

4. If the twelfth term of an A. P. is 15, and the twentieth term is 25, what is the common difference?

5. The seventh term of an A. P. is 5, and the twelfth term is 30. Find the common difference.

6. The first term of an A. P. is 7, and its third term is 13. Find the tenth term.

7. The first term of an A. P. is 20, and its sixth term is 10. Find the twelfth term.

8. The seventh term of an A. P. is 5, and the fifth term is 7. Find the twelfth term.

9. Which term of the series $5, 8, 11, \cdots$ is 65?

10. Which term of the series $\frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \cdots$ is 18?

11. Insert 6 arithmetical means between 8 and 29.

12. Insert 7 arithmetical means between 269 and 295.

13. Insert 15 arithmetical means between 67 and 43.

14. If a, b, c, d are in A.P., prove that $a + d = b + c$.

15. The sum of the second and fifth terms of an A. P. is 32, and the sum of the third and eighth is 48. Find the first term.

16. The sum of the third and fourth terms of an A. P. is 187, and the sum of the seventh and eighth terms is 147. Find the second term.

Find the sum of each of the following series:

17. $5, 9, 13, \cdots$ to 19 terms.

18. $1, 2\frac{1}{4}, 3\frac{1}{2}, \cdots$ to 12 terms.

19. $-5, -1, 3, \cdots$ to 20 terms.

20. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \cdots$ to 7 terms.

21. $10, \frac{29}{3}, \frac{28}{3}, \cdots$ to 7 terms.

22. $\frac{7}{6}, 1, \frac{5}{6}, \cdots$ to 15 terms.

How many terms must be taken of

23. The series 42, 39, 36, ⋯ to make 315?

24. The series 15, 12, 9, ⋯ to make 45?

25. The series $-8, -7, -6, \cdots$ to make 42?

26. Find the sum of all the numbers between 100 and 500 which are divisible by 3.

27. Find the sum of all the odd numbers between 100 and 200.

28. The sum of 10 terms of an arithmetical series is 145, and the sum of its fourth and ninth terms is 5 times the third term. Determine the series.

29. Divide 80 into 4 parts which are in A. P., and which are such that the product of the first and fourth parts is $\frac{2}{3}$ of the product of the second and third.

30. Find 4 numbers in A. P., such that the sum of their squares shall be 120, and that the product of the first and last shall be less than the product of the other two by 8.

31. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on; (1) how far will it fall during the tth second? (2) how far will it fall in t seconds? *Ans.* $(2t-1)a$, at^2.

32. How many strokes does a common clock make in 12 hours?

33. A debt can be discharged in a year by paying $1 the first week, $3 the second week, $5 the third, and so on. Find the last payment and the amount of the debt.

34. One hundred apples are placed on the ground at the distance of a yard from one another. How far will a person travel, who shall bring them, one by one, to a basket, placed at a distance of a yard from the first apple?

35. Two boys A and B set out at the same time, to meet each other, from two places 343 miles apart, their daily journeys being in A. P.; A's common difference being an increase of two miles, and B's a decrease of 5 miles. On the day at the end of which they met, each travelled exactly 20 miles. Find the duration of each journey.

GEOMETRICAL PROGRESSION.

318. A **Geometrical Progression** is a series in which the ratio of any term to the preceding term is the same throughout the series. This ratio is called the **Common Ratio.** It may be either positive or negative.

Thus the series,

$$2, \quad 6, \; 18, \quad 54, \; 162, \; \cdots,$$

$$8, \quad 4, \; 2, \quad 1, \; \tfrac{1}{2}, \; \cdots,$$

and

$$\tfrac{3}{2}, \; -1, \; \tfrac{2}{3}, \; -\tfrac{4}{9}, \; \tfrac{8}{27}, \; \cdots,$$

are geometric progressions (G. P.). In the first series, the common ratio is 3; in the second series it is $\frac{1}{2}$; and in the last it is $-\frac{2}{3}$.

If we multiply any term in either series by its common ratio, the product will be the next term of that series.

319. The nth Term. Let r denote the common ratio, and a the first term of a G. P.; then by definition

$$\text{the second term} = ar,$$

$$\text{the third term} = ar^2,$$

$$\text{the fifth term} = ar^4,$$

and

$$\text{the } n\text{th term} = ar^{n-1}. \qquad (1)$$

Thus, if the first term of a G. P. is 8, and the common ratio is $\frac{1}{2}$,

$$\text{the fifth term} = 8 \times (\tfrac{1}{2})^{5-1} = \tfrac{1}{2},$$

and

$$\text{the ninth term} = 8 \times (\tfrac{1}{2})^{9-1} = \tfrac{1}{32}.$$

Example. The sixth term of a G. P. is 156, and the eighth term is 7644. Find the seventh term.

Here $156 = \text{sixth term} = ar^5,$ (1)

and $7644 = \text{eighth term} = ar^7.$ (2)

Divide (2) by (1), $49 = r^2.$ (3)

$\therefore r = \pm 7.$ (4)

From (1) and (4), $a = 156 \div (\pm 7)^5.$

$$\therefore \text{the seventh term} = 156 \div (\pm 7)^5 \times (\pm 7)^6$$
$$= 156 (\pm 7) = \pm 1092.$$

320. When three numbers, a, b, c, are in G. P., the middle term b is called the **Geometric Mean** of the other two terms a and c.

321. If a, b, c are in G. P., by § 318, we have

$$c : b = b : a. \quad \therefore b = \sqrt{ac}.$$

That is, *the geometric mean of any two numbers is the mean proportional between them.*

322. All the terms between any two terms of a G. P. may be called the **Geometric Means** of the two terms.

323. *To insert* m *geometric means between* a *and* b.

Calling a the first term, b will be the $(m+2)$th term; hence by (1) of § 319, we have

$$b = ar^{m+1}.$$
$$\therefore r = \sqrt[m+1]{b \div a}. \quad (1)$$

Hence the m means required are ar, ar^2, $\cdots ar^m$, in which r has the value given in (1).

Example. Insert 6 geometric means between 56 and $-\frac{7}{16}$.

Here $a = 56,\ b = -\frac{7}{16},$ and $m + 1 = 7.$

$$\therefore r = \sqrt[7]{-\frac{7}{16} \div 56} = \sqrt[7]{-\frac{1}{2^4 \times 2^3}} = -\frac{1}{2}.$$

Hence $r = -\frac{1}{2}$, and the 6 means required are

$$-28,\ 14,\ -7,\ \tfrac{7}{2},\ -\tfrac{7}{4},\ \tfrac{7}{8}.$$

324. Sum of n Terms. Let s denote the sum of n terms; then

$$\begin{aligned} s &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \\ &= a(1 + r + r^2 + r^3 + \cdots + r^{n-2} + r^{n-1}) \\ &= a\frac{1-r^n}{1-r}. \qquad \S\ 142 \end{aligned}$$

Hence $$s = \frac{a(1-r^n)}{1-r}. \qquad (1)$$

Let l denote the nth term; then from § 319

$$l = ar^{n-1}. \qquad (2)$$

From (1) and (2)

$$s = \frac{a - rl}{1-r}. \qquad (3)$$

If any three of the five numbers a, l, n, r, s, be known, the other two may be found from (1) and (2), or from (3) and (2).

Example. Sum the series 6, -18, 54, $\cdots$ to 6 terms.

Here $a = 6$, $r = -18 \div 6 = -3$, $n = 6$.

From (1),
$$\begin{aligned} s &= \frac{6\{1-(-3)^6\}}{1-(-3)} \\ &= \tfrac{3}{2}\{1 - 3^6\} \\ &= -1092. \end{aligned}$$

325. When $r<1$ numerically, the successive terms of a G. P. become smaller and smaller numerically, and the G. P. is said to be a *decreasing progression*.

Thus if $a = 2$ and $r = \frac{1}{2}$, we have the decreasing G. P.,

$$2,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \tfrac{1}{32},\ \tfrac{1}{64},\ \tfrac{1}{128},\ \tfrac{1}{256},\ \cdots. \qquad (1)$$

The sum of an infinite number of terms of this series approaches 4 as its limit. For suppose that we bisect a line four inches long, and take away one of the parts; then bisect the remainder, and take away one of the parts; and continue this process without limit. It is evident that the part remaining will approach zero as its limit, and the sum of the successive parts taken away will approach four inches as its limit. But the lengths in inches of the successive parts taken away will be the terms of series (1). Hence the sum of an infinite number of terms of that series approaches 4 as its limit.

326. *The limit of the sum of an infinite number of terms of a decreasing G. P. is* $\frac{a}{1-r}$.

For, from (1) of § 324, we have

$$s = \frac{a}{1-r} - \frac{ar^n}{1-r}. \qquad (1)$$

Now if $r < 1$ numerically, and the number of terms, or n, be increased without limit, then

$$r^n \doteq 0. \quad \therefore \frac{ar^n}{1-r} \doteq 0.$$

Hence from (1), by § 281, we obtain

$$\text{lt}(s) = \frac{a}{1-r}. \qquad (2)$$

The *limit of the sum* of an infinite number of terms of a series is often called the *sum*.

EXAMPLE 1. Sum the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\cdots$, to infinity.

Here $a = 1$, $r = \frac{1}{2}$.

From (2), $\text{lt}(s) = \frac{1}{1-\frac{1}{2}} = 2.$

EXAMPLE 2. Sum the series 9, -3, 1, $\cdots$, to infinity.

Here $a = 9$, $r = -\frac{1}{3}$.

From (2), $\text{lt}(s) = \frac{9}{1-(-\frac{1}{3})} = \frac{27}{4} = 6\frac{3}{4}.$

Example 3. Find the value of $0.4\dot{2}\dot{3}$.

$$0.4\dot{2}\dot{3} = 0.4232323\cdots = \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \cdots.$$

Now,
$$\frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} + \cdots = \frac{23}{10^3} \div \left(1 - \frac{1}{10^2}\right)$$
$$= \frac{23}{10^3} \times \frac{10^2}{99} = \frac{23}{990}.$$
$$\therefore\ 0.4\dot{2}\dot{3} = \tfrac{4}{10} + \tfrac{23}{990} = \tfrac{419}{990}.$$

Example 4. Find the G. P. whose sum to infinity is 18, and whose second term is − 8.

Here
$$ar = -8, \qquad (1)$$
and
$$\frac{a}{1-r} = 18. \qquad (2)$$

Divide (1) by (2), $r(1-r) = -\tfrac{4}{9}$.
$$\therefore\ r^2 - r - \tfrac{4}{9} = 0.$$
$$\therefore\ r = -\tfrac{1}{3}, \text{ or } \tfrac{4}{3}.$$

Only the value $-\tfrac{1}{3}$ is admissible for r, since the series is a decreasing one.

From (1), $a = -8 \div (-\tfrac{1}{3}) = 24.$

Hence the series is $24,\ -8,\ \tfrac{8}{3},\ -\tfrac{8}{9},\ \cdots$.

Exercise 97.

Find the last term in the following series: —

1. $2, 4, 8, \cdots$, to 9 terms. **2.** $2, 3, 4\tfrac{1}{2}, \cdots$, to 6 terms.

3. $3, -3^2, 3^3, \ldots$, to $2n$ terms.

4. $x, 1, \dfrac{1}{x}, \cdots$, to 30 terms.

5. The first term of a G. P. is 3, and the third term is 4. Find the fifth term.

6. The third term of a G. P. is 1, and the sixth term is $-\tfrac{1}{8}$. Find the tenth term.

7. The fourth term of a G. P. is 0.016, and the seventh term is 0.000128. Find the first term.

8. The fourth term of a G. P. is $\frac{1}{18}$, and the seventh term is $-\frac{1}{486}$. Find the sixth term.

9. Insert 3 geometric means between 486 and 6.

10. Insert 4 geometric means between $\frac{1}{8}$ and 128.

11. Insert 5 geometric means between 3 and 0.000192.

12. Insert 4 geometric means between a^3b^{-6} and $a^{-2}b^4$.

Find the sum of the series

13. 64, 32, 16, ⋯ to 10 terms.

14. 8.1, 2.7, 0.9, ⋯ to 7 terms.

15. $3, -1, \frac{1}{3}, \cdots$ to 6 terms.

16. $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \cdots$ to 7 terms.

17. $-\frac{2}{5}, \frac{1}{2}, -\frac{5}{8}, \cdots$ to 6 terms.

18. $2, -4, 8, \cdots$ to $2p$ terms.

Sum to infinity the following series:

19. 9, 6, 4, ⋯

20. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \cdots$

21. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \cdots$

22. $\frac{8}{5}, -1, \frac{5}{8}, \cdots$

23. 0.9, 0.03, 0.001, ⋯

24. 0.8, −0.4, 0.2, ⋯

By the method of § 326 find the value of

25. $0.\dot{3}$. **26.** $0.1\dot{6}$. **27.** $0.\dot{2}\dot{4}$. **28.** $0.3\dot{7}\dot{8}$. **29.** $0.\dot{0}3\dot{7}$.

30. Find the G. P. whose sum to infinity is 4, and whose second term is $\frac{3}{4}$.

31. Find the G. P. whose sum to infinity is 9, and whose second term is −4.

32. If every alternate term of a G. P. be taken away, the remaining terms will be in G. P.

33. If all the terms of a G. P. be multiplied by the same number, the products will be in G. P.

34. Show that the reciprocals of the terms of a G. P. are in G. P.

35. By saving 1 cent the first day, 2 cents the second day, 4 cents the third day, and so on, doubling the amount every day, how much would be saved in a month of 30 days?

36. Suppose a body to move eternally as follows: 20 feet the first minute, 19 feet the second minute, $18\frac{1}{20}$ feet the third minute, and so on. Find the limit of the distance passed over.

37. A ball falls from the height of 100 feet, and at every fall rebounds one fourth the distance. Find the distance passed through by the ball before it comes to rest.

38. If in Problem 31 of Exercise 96, $a = 16\frac{1}{12}$, find how long it will be before the ball in the preceding problem comes to rest.

To fall 100 feet, it takes $\sqrt{100 \div 16\frac{1}{12}}$, or $10\sqrt{\frac{12}{193}}$, seconds; to rebound, or to fall, 25 feet, it takes $\sqrt{25 \div 16\frac{1}{12}}$, or $5\sqrt{\frac{12}{193}}$, seconds; to rebound, or to fall, $6\frac{1}{4}$ feet, it takes $\sqrt{6\frac{1}{4} \div 16\frac{1}{12}}$, or $\frac{5}{2}\sqrt{\frac{12}{193}}$, seconds; and so on.

$$\text{Hence the time } = 10\sqrt{\tfrac{12}{193}} + 2\left(5\sqrt{\tfrac{12}{193}} + \tfrac{5}{2}\sqrt{\tfrac{12}{193}} + \cdots\right)$$
$$= 30\sqrt{\tfrac{12}{193}} = \tfrac{60}{193}\sqrt{579} = 7.4805+$$

HARMONIC PROGRESSION.

327. An **Harmonic Progression** is a series of numbers whose reciprocals form an A. P.

Thus the two series

$$1,\ \tfrac{1}{3},\ \tfrac{1}{5},\ \tfrac{1}{7},\ \cdots, \text{ and } 4,\ -4,\ -\tfrac{4}{3},\ \cdots$$

are each an harmonic progression; for the reciprocals of their terms

$$1,\ 3,\ 5,\ 7,\ \cdots, \text{ and } \tfrac{1}{4},\ -\tfrac{1}{4},\ -\tfrac{3}{4},\ \cdots$$

are in A. P.

328. When three numbers are in harmonic progression (H. P.), the middle term is called the **Harmonic Mean** of the other two.

329. Let H be the harmonic mean of a and b; then by § 327,

$$\frac{1}{a},\ \frac{1}{H},\ \frac{1}{b}\ \text{are in A. P.}$$

$$\therefore\ \frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H}.$$

$$\therefore\ \frac{2}{H}=\frac{1}{a}+\frac{1}{b},\ \text{or}\ H=\frac{2ab}{a+b}.$$

330. If A and G denote respectively the arithmetical and the geometric mean of a and b, then (§§ 315, 321)

$$A=\frac{a+b}{2},\ G=\sqrt{ab},\ H=\frac{2ab}{a+b}.$$

$$\therefore\ A\times H=\frac{a+b}{2}\times\frac{2ab}{a+b}=ab=G^2.$$

Hence $\qquad A:G=G:H.$

That is, *the geometric mean of any two numbers is also the geometric mean of their arithmetic and harmonic means.*

331. Problems in H. P. are generally solved by inverting the terms, and making use of the properties of the resulting A. P.

EXAMPLE. The fifteenth term of an H. P. is $\frac{1}{25}$, and the twenty-third term is $\frac{1}{41}$. Find the series.

Let a be the first term, and d the common difference of the corresponding A. P.; then

$$25 = \text{the fifteenth term} = a + 14\,d,$$

and $$41 = \text{the twenty-third term} = a + 22\,d.$$

$$\therefore\ d = 2,\ a = -3.$$

Hence the A. P. is $-3, -1, 1, 3, 5, \cdots,$

and the H. P. is $-\frac{1}{3}, -1, 1, \frac{1}{3}, \frac{1}{5} \cdots.$

Exercise 98.

1. Find the sixth term of the series $4, 2, 1\frac{1}{3}, \cdots.$

2. Find the eighth term of the series $1\frac{1}{3}, 1\frac{11}{17}, 2\frac{2}{13}, \cdots.$

Kind the series in which

3. The second term is 2, and the thirty-first term is $\frac{4}{81}$.

4. The thirty-ninth term is $\frac{1}{11}$, and the fifty-fourth term is $\frac{1}{26}$.

Find the harmonic mean between

5. 2 and 4. 6. 1 and 13. 7. $\frac{1}{4}$ and $\frac{1}{10}$.

8. Insert 2 harmonic means between 4 and 12.

9. Insert 3 harmonic means between $2\frac{2}{5}$ and 12.

10. Insert 4 harmonic means between 1 and 6.

11. If a, b, c, are in harmonic progression, prove that $a - b : b - c = a : c$.

CHAPTER XXVI.

LOGARITHMS.

332. The **Logarithm** of a number is the exponent by which a fixed *base* must be affected in order to be equal to the number. That is, if $a^x = N$, x is the logarithm of N to the base a, which is written $x = \log_a N$.

Thus, since $3^2 = 9$, $2 = \log_3 9$.

Since $2^4 = 16$, $4 = \log_2 16$.

Since $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $\cdots$,

the positive numbers 1, 2, 3, $\cdots$, are respectively the logarithms of 10, 100, 1000, $\cdots$, to the base 10.

To the base 10 the logarithms of all numbers between 1 and 10, 10 and 100, 100 and 1000, $\cdots$, are incommensurable.

Since $3^{-2} = \frac{1}{9}$, $-2 = \log_3 \frac{1}{9}$.

Since $10^{-1} = 0.1$, $10^{-2} = 0.01$, $10^{-3} = 0.001$, $\cdots$,

the negative numbers -1, -2, -3, $\cdots$, are respectively the logarithms of 0.1, 0.01, 0.001, $\cdots$, to the base 10.

333. Any positive number except 1 may evidently be taken as the base of logarithms. The following are the *fundamental properties* of logarithms to any base.

334. *The logarithm of* 1 *is zero.*

For $a^0 = 1$. $\therefore \log_a 1 = 0$.

335. *The logarithm of the base itself is* 1.

For $a^1 = a$. $\therefore \log_a a = 1$.

336. *The logarithm of a product equals the sum of the logarithms of its factors.*

For let $M = a^x,\ N = a^y;$

then $M \times N = a^{x+y}.$

Hence $\log_a(M \times N) = x + y = \log_a M + \log_a N.$

337. *The logarithm of a quotient equals the logarithm of the dividend minus that of the divisor.*

For let $M = a^x,\ N = a^y;$

then $M \div N = a^{x-y}.$

Hence $\log_a(M \div N) = x - y = \log_a M - \log_a N.$

338. *The logarithm of a positive number affected with any exponent equals the logarithm of the number multiplied by the exponent.*

Let $M = a^x;$

then, whatever be the value of p,

$$M^p = a^{px}.$$

Hence $\log_a(M^p) = px = p \log_a M.$

339. By § 338, the logarithm of any power of a number equals the logarithm of the number multiplied by the exponent of the power; and the logarithm of any root of a number equals the logarithm of the number divided by the index of the root.

340. From the principles proved above, we see that by the use of logarithms the operations of multiplication and division may be replaced by those of addition and subtraction, and the operations of involution and evolution by those of multiplication and division.

Example. Find $\log_a \dfrac{\sqrt[3]{x^2}}{b^3c^{\frac{5}{4}}}$ in terms of $\log_a x$, $\log_a b$, $\log_a c$.

$$\log_a \frac{\sqrt[3]{x^2}}{b^3c^{\frac{5}{4}}} = \log_a x^{\frac{2}{3}} - \log_a (b^3c^{\frac{5}{4}}) \qquad \S\ 336$$

$$= \log_a x^{\frac{2}{3}} - (\log_a b^3 + \log_a c^{\frac{5}{4}}) \qquad \S\ 335$$

$$= \tfrac{2}{3} \log_a x - 3 \log_a b - \tfrac{5}{4} \log_a c. \qquad \S\ 337$$

Exercise 99.

1. Find $\log_2 8$; $\log_4 64$; $\log_3 81$; $\log_2 32$; $\log_5 125$; $\log_4 \frac{1}{16}$; $\log_9 \frac{1}{81}$; $\log_3 \frac{1}{81}$; $\log_2 \frac{1}{32}$; $\log_3 \frac{1}{27}$; $\log_5 \frac{1}{125}$.

2. If 10 is the base, between what integral numbers does the logarithm of any number between 1 and 10 lie? Of any number between 10 and 100? Of any number between 100 and 1000? Of any number between 0.1 and 1? Of any number between 0.01 and 0.1? Of any number between 0.001 and 0.01?

In the next six examples express $\log_a y$ in terms of $\log_a b$, $\log_a c$, $\log_a x$, and $\log_a z$.

3. $y = x^3b^{\frac{2}{3}}c^5.$

4. $y = \sqrt[3]{z^2} \cdot \sqrt[5]{c^6}.$

5. $y = \dfrac{b^3x^2z}{c^{\frac{2}{3}}}.$

6. $y = \sqrt[7]{b^2x^4z^3}.$

7. $y = \sqrt{xz^5} \cdot \sqrt[3]{b^2c^7}.$

8. $y = \dfrac{\sqrt{x^3zb^5}}{\sqrt[3]{x^2z^4c^5}}.$

COMMON LOGARITHMS.

341. The logarithms used for numerical computations are always those to base 10; for this reason logarithms to base 10 are called **Common Logarithms.** Hereafter in this chapter when no base is written, the base 10 is to be understood. It is evident that the logarithms of most numbers consist of an integral part and a fractional part.

The common logarithm of a number less than 1 is negative; but for convenience it is always written in such a form that the *fractional part* will be *positive*.

Thus the common logarithm of any number between

1 and 10 will be $0 +$ a fraction,
10 and 100 will be $1 +$ a fraction,
100 and 1000 will be $2 +$ a fraction,
0.1 and 1 will be $-1 +$ a fraction,
0.01 and 0.1 will be $-2 +$ a fraction,
0.001 and 0.01 will be $-3 +$ a fraction.

342. The integral part of a logarithm is called the **Characteristic**; and the fractional part, which is always *positive*, the **Mantissa.**

Thus in the logarithm 2.5798, 2 is the characteristic and .5798 the mantissa. In the logarithm $-2 + .7168$, which is often written $\bar{2}.7168$, -2 is the characteristic and .7168 the mantissa.

343. *If two numbers differ only in the position of the decimal point, their common logarithms have the same mantissa.*

For $$\log(N \times 10^m) = \log N + m.$$

Now if m is any whole number either positive or negative, the numbers N and $N \times 10^m$ will differ only in the position of the decimal point; and $\log N$ and $\log(N \times 10^m)$ will differ only by the whole number m; that is, they will have the same mantissa.

Thus having given that $\log 31 = 1.4914$, we have

$$\begin{aligned}\log 0.031 = \log(31 \div 1000) &= \log 31 - \log 1000\\ &= 1.4914 - 3\\ &= 8.4914 - 10.\end{aligned}$$

Here the negative characteristic -2 is written in the very common form $8-10$.

344. The *characteristic* of the common logarithm of any number can be determined by one of the two following simple rules:

(i.) *If the number is greater than unity, the characteristic is positive and numerically one less than the number of digits in its integral part.*

For let N denote a number that has n digits in its integral part; then N lies between 10^{n-1} and 10^{n}; that is,

$$N = 10^{(n-1)+\text{a fraction}}.$$

$$\therefore \log N = (n-1) + \text{a mantissa}.$$

Thus 785 lies between 10^2 and 10^3;

hence $\log 785 = 2 + \text{a mantissa}$.

(ii.) *If the number is less than unity, the characteristic is negative and numerically one greater than the number of ciphers immediately after the decimal point.*

For let N denote a decimal with n ciphers immediately after the decimal point; then N lies between $10^{-(n+1)}$ and 10^{-n}; that is,

$$N = 10^{-(n+1)+\text{a fraction}}.$$

$$\therefore \log N = -(n+1) + \text{a mantissa}.$$

Thus 0.0078 lies between 10^{-3} and 10^{-2};

hence $\log 0.0078 = -3 + \text{a mantissa}$.

The converse of rules (i.) and (ii.) may be stated as follows:

(i.) *If the characteristic of a common logarithm is* $+n$, *there are* $n+1$ *integral places in the corresponding number.*

(ii.) *If the characteristic is* $-n$, *there are* $n-1$ *ciphers immediately to the right of the decimal point in the number.*

Thus if $\log N = 3.6712$, then the number N will have four integral places. If $\log N = \bar{4}.9345$, then N will be less than 1, and there will be three ciphers immediately to the right of the decimal point in the number N.

345. When a negative logarithm is to be divided by any number, the logarithm must be so modified in form that the negative part will be exactly divisible by the number.

Example. Given $\log 0.0785 = \bar{2}.8949$, find $\log \sqrt[7]{0.0785}$.

$$\begin{aligned}\text{Log}\ \sqrt[7]{0.0785} &= \tfrac{1}{7}\log 0.0785.\\ &= \tfrac{1}{7}(\bar{2}.8949)\\ &= \tfrac{1}{7}(\bar{7}. + 5.8949) = \bar{1}.8421.\end{aligned}$$

Adding both -5 and $+5$ to the logarithm does not change its value and makes its negative part divisible by 7.

Exercise 100.

Given $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$; $\log 11 = 1.0414$; find the common logarithm of

1. 0.0105.

$$\begin{aligned}105 &= 3 \times 5 \times 7.\\ \therefore \log 105 &= \log 3 + \log 5 + \log 7\\ &= 0.4771 + 0.6990 + 0.8451\\ &= 2.0212.\\ \therefore \log 0.0105 &= \bar{2}.0212 \text{ or } 8.0212 - 10. \qquad \S\ 343\end{aligned}$$

2. 15.
3. 21.
4. 56.
5. 112.
6. 216.
7. 147.
8. 34.3.
9. 37.5.
10. 840.
11. 8.4.
12. 0.128.
13. 0.0035.
14. 0.00021.
15. 0.0216.
16. 0.0054.
17. 0.0063.
18. $77 \div 15$.
19. $21 \div 16$.
20. $42 \div 25$.
21. $7\frac{1}{7}$.
22. 21^5.
23. $5^{\frac{1}{5}}$.
24. $12^{\frac{2}{3}}$.
25. $15^{\frac{5}{6}}$.
26. $\sqrt[9]{98}$.
27. $\sqrt[12]{126}$.
28. $\sqrt{0.231}$.
29. $\sqrt[3]{0.048}$.
30. $\sqrt[4]{0.0054}$.
31. $\sqrt[6]{0.00021}$.
32. $\sqrt[5]{0.0105}$.
33. $\sqrt[7]{0.0165}$.
34. $\sqrt[11]{0.0693}$.
35. $(15 \div 14)^{\frac{11}{7}}$.
36. $(0.21 \div 0.11)^{\frac{1}{5}}$.
37. $(0.08 \div 0.03)^{\frac{2}{3}}$.

346. Tables of Logarithms. Common logarithms have two great practical advantages: (i.) Characteristics are known by § 344, so that only mantissas are tabulated; (ii.) mantissas are determined by the sequence of digits (§ 343), so that the mantissas of integral numbers only are tabulated.

At the close of this chapter will be found a table which contains the *mantissas* of the common logarithms of all numbers from 1 to 999 correct to four decimal places.

NOTE. Tables are published which give the logarithms of all numbers from 1 to 99999 calculated to seven places of decimals; these are called 'seven-place' logarithms. For many purposes, however, the four-place or five-place logarithms are sufficiently accurate.

From a table of logarithms we may obtain (i.) the logarithm of a given number, or (ii.) the number corresponding to a given logarithm.

347. *To find the logarithm of a given number.*

(1) To find log 7.85.

By § 343 the required mantissa is the mantissa of log 785.

Look in column headed "N" for 78. Passing along this line to column headed 5, we find .8949, the required mantissa.

Adding the characteristic, we have

$$\log 7.85 = 0.8949.$$

(2) To find log 4273.2.

When the number contains more than three significant figures, use must be made of the principle that when the difference of two numbers is *small* compared with either of them, the difference of the numbers is approximately proportional to the difference of their logarithms.

By § 343 the required mantissa is that of log 427.32.

$$\text{The mantissa of } \log 427 = .6304.$$
$$\text{The mantissa of } \log 428 = .6314.$$

That is, an increase of 1 in the number causes an increase of .0010 in the mantissa; hence an increase of .32 in the number will cause an increase of .32 of .0010, or .0003, in the mantissa. Adding .0003 to the mantissa of log 427, and writing the characteristic, we have log 4273.2 = 3.6307.

(3) To find log 0.0006049.

By § 343 the required mantissa is that of log 604.9.

The mantissa of log 604 = .7810.

The mantissa of log 605 = .7818.

Hence .9 of .0008, or .0007, must be added to .7810.

$\therefore$ log 0.0006049 = $\bar{4}.7817$, or 6.7817 − 10.

(4) To find log 30 or log 3, find mantissa of log 300.

348. *To find a number when its logarithm is given.*

(1) To find the number of which the logarithm is 3.8954.

Look in the table for the mantissa .8954. It is found in line 78 and in column 6; hence .8954 is the mantissa of log 786.

$\therefore$ 3.8954 = log 7860;

or 7860 is the number whose logarithm is 3.8954.

(2) To find the number of which the logarithm is 1.6290.

Look in the table for the mantissa .6290. It cannot be found; but the next less mantissa is 6284, and the next greater is 6294.

Also, 6284 = mantissa of log 425,

and 6294 = mantissa of log 426.

That is, an increase of .0010 in the mantissa causes an increase of 1 in the number; hence an increase of .0006 in the mantissa will cause an increase of $\frac{6}{10}$ of 1, or .6, in the number; hence .6290 is the mantissa of log 425.6; and therefore 1.6290 = log 42.56.

(3) To find the number of which the logarithm is $\bar{3}.8418$.

Look in the table for the mantissa 8418. It cannot be found;

but 8414 = mantissa of log 694,

and 8420 = mantissa of log 695.

That is, an increase of .0006 in the mantissa causes an increase of 1 in the number; hence an increase of .0004 in the mantissa will cause an increase of $\frac{4}{6}$ of 1, or .66, in the number; hence 8418 is the mantissa of log 694.66.

$$\therefore \bar{3}.8418 = \log 0.0069466.$$

Exercise 101.

Find, from the table, the logarithm of

1. 8.
2. 50.
3. 6.3.
4. 374.
5. 703.
6. 7.89.
7. 0.178.
8. 3.476.
9. 0.05307.
10. 78542.
11. 0.50438.
12. 0.00716.
13. 7.4803.
14. 2063.4.
15. 0.0087741.
16. 0.017423.

Find the number of which the logarithm is

17. 1.8797.
18. 7.6284 – 10.
19. 0.2165.
20. 2.7364.
21. 4.0095.
22. 8.1648 – 10.
23. 9.3178 – 10.
24. 1.6482.
25. 8.5209 – 10.
26. 3.8016.
27. 3.7425.
28. 7.1342 – 10.
29. 3.7045.
30. 8.7982 – 10.
31. 3.4793.

349. The **Cologarithm** of a number is the logarithm of its reciprocal.

That is, $\text{colog}\, N = \log(1 \div N) = -\log N$

$$= (10 - \log N) - 10.$$

If $\log N > 0$ and < 10, $\text{colog}\, N$ is written

$$(10 - \log N) - 10,$$

to avoid a negative decimal in the cologarithm.

If $\log N > 10$ and < 20, $\text{colog}\, N$ is written

$$(20 - \log N) - 20,$$

for the same reason.

$$\begin{aligned} \text{Thus}\quad \text{colog } 0.0574 &= -(\bar{2}.7589) = 1.2411;\\ \text{colog } 432 &= (10 - 2.6263) - 10 = 7.3737 - 10;\\ \text{colog } 345000000000 &= (20 - 11.5378) - 20 = 8.4622 - 20. \end{aligned}$$

Instead of subtracting the logarithm of a divisor, we may by § 73 add its cologarithm.

Example. Find the value of $\dfrac{15.08 \times 0.0723}{0.0534 \times 7.238}$.

$$\begin{aligned} \log\ 15.08 &= 1.1784\\ \log 0.0723 &= 8.8591 - 10\\ \text{colog } 0.0534 &= 1.2725\\ \text{colog }\ 7.238 &= \underline{9.1404 - 10} \end{aligned}$$

$$\begin{aligned} &\text{Add,}\quad \log\text{ (fraction)} = 0.4504 \qquad = \log 2.8213.\\ &\text{Hence}\quad \text{the fraction} = 2.8213. \end{aligned}$$

Example 2. Find the value of $0.0543 \times 6.34 \times (-5.178)$.

$$\begin{aligned} \log 0.0543 &= 8.7348 - 10\\ \log\ 6.34 &= 0.8021\\ \log\ 5.178 &= \underline{0.7141\qquad} \end{aligned}$$

$$\text{Add,}\quad \log\text{ (product)} = 0.2510 \qquad = \log 1.7824.$$

Hence the product is -1.7824.

Note. By logarithms we obtain simply the arithmetical value of the result; its quality must be determined by the laws of quality.

Example 3. Find the value of $\sqrt[5]{\dfrac{5.42 \times 427.^2}{3.24^4 \times 0.0231^{\frac{1}{2}}}}$.

$$\begin{aligned} \log 5.42 &= 0.7340 & &= 0.7340\\ 2 \log 427. &= (2.6304) \times 2 & &= 5.2608\\ 4 \text{ colog } 3.24 &= (9.4895 - 10) \times 4 & &= 7.9580 - 10\\ \tfrac{1}{2} \text{ colog } 0.0231 &= (1.6364) \div 2 & &= \underline{0.8182\qquad\quad}\\ & & &5)\underline{4.7710}\\ & & \therefore \log\text{ (root)} &= 0.9542\\ & & \therefore \text{root} &= 9. \end{aligned}$$

350. An **Exponential Equation** is one in which the unknown letter appears in an exponent; as $2^x = 5$, $x^x = 10$. Such equations are solved by the aid of logarithms.

Example 1. Solve $\qquad 3^{2x} - 14 \times 3^x + 45 = 0. \qquad (1)$

From (1), $\qquad (3^x - 9)(3^x - 5) = 0. \qquad (2)$

Equation (2) is equivalent to the two equations

$$3^x = 9 \text{ and } 3^x = 5.$$

From $\qquad 3\ \ = 9$, $x = 2$; and from $3^x = 5$,

$$x \log 3 = \log 5.$$

$$\therefore x = \frac{\log 5}{\log 3} = \frac{0.6990}{0.4771} = 1.4649.$$

Hence the solutions of (1) are 2 and 1.4649.

Exercise 102.

Find by logarithms the value of

1. 742.8×0.02374.
2. 0.3527×0.00572.
3. 78.42×0.000437.
4. $5234 \times (-0.03671)$.
5. $3.246 \times (-0.0746)$.
6. $-4.278 \times (-0.357)$.
7. $4743 \div 327.4$.
8. $9.345 \div (-0.0765)$.
9. $\dfrac{2.476 \times (-0.742)}{73.81 \times (-0.00121)}$.
10. $\dfrac{321 \times (-48.1) \times (357)}{421 \times (-741) \times (4.21)}$.
11. $5^{\frac{2}{3}}$.
12. $0.021^{\frac{5}{2}}$.
13. 0.532^8.
14. $(\frac{17}{54})^9$.
15. $714.2^{\frac{2}{5}}$.
16. $(\frac{471}{891})^7$.
17. $(\frac{15}{23})^{2.7}$.
18. $(3\frac{2}{5})^{1.27}$.
19. $4.71^{3.205}$.
20. $\sqrt{\dfrac{0.035^3 \times 54.2 \times 785^{\frac{1}{4}} \times 0.0742}{4.72^{\frac{1}{3}} \times 7.14^{\frac{1}{5}} \times 8.47^{\frac{1}{6}}}}$.

21. $\sqrt[3]{\dfrac{0.0427^2 \times 5.27 \times 0.875^4}{7.421^{\frac{1}{4}} \times \sqrt{1.74} \times \sqrt{0.00215}}}.$

22. $\sqrt[5]{\dfrac{0.714^{\frac{1}{2}} \times 0.1371^{\frac{1}{3}} \times 0.0718^{\frac{1}{4}}}{0.524^2 \times 0.742^{\frac{1}{3}} \times 0.0527^{\frac{1}{2}}}}.$

Solve

23. $31^x = 23.$ 25. $5^x = 800.$ 27. $5^{x-3} = 8^{2x+1}.$

24. $0.3^x = 0.8.$ 26. $12^x = 3528.$ 28. $a^{2x}b^{3x} = c^5.$

29. $2^{3x}5^{2x-1} = 4^{5x}3^{x+1}.$ 30. $4^{2x} + 56 = 15 \times 4^x.$

COMPOUND INTEREST AND ANNUITIES.

351. *To find the compound interest,* \$ I, *and amount,* \$ M, *of a given principal,* \$ P, *in* n *years,* \$ r *being the interest on* \$ 1 *for* 1 *year.*

Let \$ R = the amount of \$ 1 in 1 year; then $R = 1 + r$, and the amount of \$ P at the end of the first year is \$ PR; and since this is the principal for the second year, the amount at the end of the second year is \$ $PR \times R$, or \$ PR^2. For like reason the amount at the end of the third year is \$ PR^3, and so on; hence the amount in n years is \$ PR^n; that is,

$$M = PR^n, \text{ or } P(1 + r)^n. \tag{1}$$

Hence

$$I = P(R^n - 1). \tag{2}$$

If the interest is payable semi-annually, the amount of \$ P in $\frac{1}{2}$ a year will be \$ $P(1 + \frac{1}{2}r)$; hence, as n years equals $2n$ half-years,

$$M = P(1 + \tfrac{1}{2}r)^{2n}. \tag{3}$$

Similarly, if the interest is payable quarterly,

$$M = P(1 + \tfrac{1}{4}r)^{4n}. \tag{4}$$

Example. Find the time in which a sum of money will double itself at ten per cent compound interest, interest to be "converted into principal" semi-annually.

Here $1 + \frac{1}{2} r = 1.05$. Let $P = 1$; then $M = 2$.

Substituting these values in (3), we obtain

$$2 = (1.05)^{2n}.$$

$$\therefore \log 2 = 2n \cdot \log 1.05.$$

$$\therefore n = \frac{\log 2}{2 \log 1.05} = \frac{0.3010}{0.0424} = 7.1 \text{ years. } Ans.$$

352. Present Value and Discount. Let \$ P denote the present value of the sum \$ M due in n years, at the rate \$ r; then evidently in n years at the rate \$ r, \$ P will amount to \$ M; hence

$$M = PR^n, \text{ or } P = MR^{-n}.$$

Let \$ D be the discount; then

$$D = M - P = M(1 - R^{-n}).$$

353. An **Annuity** is a fixed sum of money that is payable once a year, or at more frequent regular intervals, under certain stated conditions. An *Annuity Certain* is one payable for a fixed number of years. A *Life Annuity* is one payable during the lifetime of a person. A *Perpetual Annuity*, or *Perpetuity*, is one that is to continue forever, as, for instance, the rent of a freehold estate.

354. *To find the amount of an annuity left unpaid for a given number of years, allowing compound interest.*

Let \$ A be the annuity, n the number of years, \$ R the amount of one dollar in one year, \$ M the required amount. Then evidently the number of dollars due at the end of the

$$\begin{aligned}\text{First year} &= A;\\ \text{Second year} &= AR + A;\\ \text{Third year} &= AR^2 + AR + A;\\ n\text{th year} &= AR^{n-1} + AR^{n-2} + \cdots + AR + A\\ &= \frac{A(R^n - 1)}{R - 1}.\end{aligned}$$

That is, $$M = \frac{A}{r}(R^n - 1). \qquad (1)$$

Example 1. Find the amount of an annuity of \$100 in 20 years, allowing compound interest at $4\frac{1}{2}$ per cent.

$$M = \frac{A}{r}(R^n - 1) = \frac{100\,(1.045^{20} - 1)}{0.045}.$$

By logarithms $$1.045^{20} = 2.4117.$$

$$\therefore M = \frac{141.17}{0.045} = 3137.11.$$

Hence the amount of the annuity is \$3137.11.

Example 2. What sum must be set aside annually that it may amount to \$50,000 in 10 years at 6 per cent compound interest?

From (1), $$A = \frac{Mr}{R^n - 1} = \frac{50,000 \times 0.06}{1.06^{10} - 1} = 3793.37.$$

Hence the required sum is \$3793.37.

355. *To find the present value of an annuity of* \$A *payable at the end of each of* n *successive years.*

Let \$$P$ denote the present value; then the amount of \$$P$ in n years will equal the amount of the annuity in the same time; that is,

$$PR^n = A(R^n - 1)r^{-1}. \qquad (1)$$

$$\therefore P = A(1 - R^{-n})r^{-1}. \qquad (2)$$

If the annuity be *perpetual*, then $n = \infty$, $R^{-n} \doteq 0$, and (2) becomes

$$P = Ar^{-1}.$$

Exercise 103.

1. Write out the logarithmic equations for finding each of the four numbers M, R, P, n.

2. In what time, at 5 per cent compound interest, will \$100 amount to \$1000?

3. Find the time in which a sum will double itself at 4 per cent compound interest.

4. Find in how many years \$1000 will become \$2500 at 10 per cent compound interest.

5. Find the present value of \$10,000 due 8 years hence at 5 per cent compound interest.

6. Find the amount of \$1 at 5 per cent compound interest in a century.

7. Show that money will increase more than seventeen-thousand-fold in a century at 10 per cent compound interest.

8. If A leaves B \$1000 a year to accumulate for 3 years at 4 per cent compound interest, find what amount B should receive.

9. Find the present value of the legacy in Example 8.

10. Find the present value, at 5 per cent, of an estate of \$1000 a year to be entered on immediately.

11. A freehold estate worth \$120 a year is sold for \$4000; find the rate of interest.

12. A man has a capital of \$20,000, for which he receives interest at 5 per cent; if he spends \$1800 every year, show that he will be ruined before the end of the 17th year.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1950	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	7857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8388	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
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